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THE

FIRST THREE SECTIONS

OF

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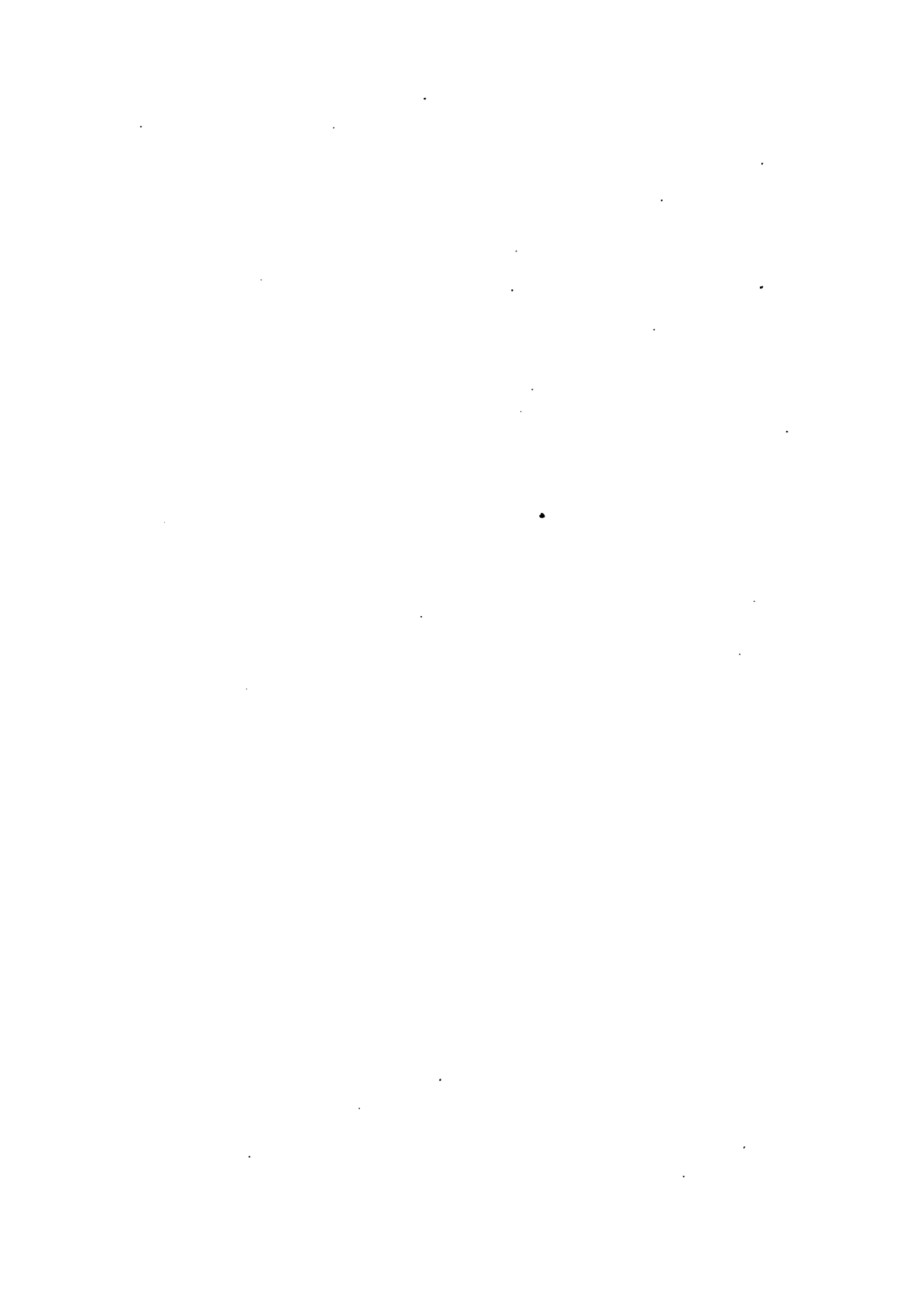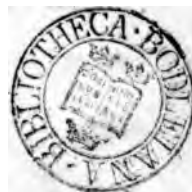
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THE

FIRST THREE SECTIONS

OF

NEWTON'S PRINCIPIA.

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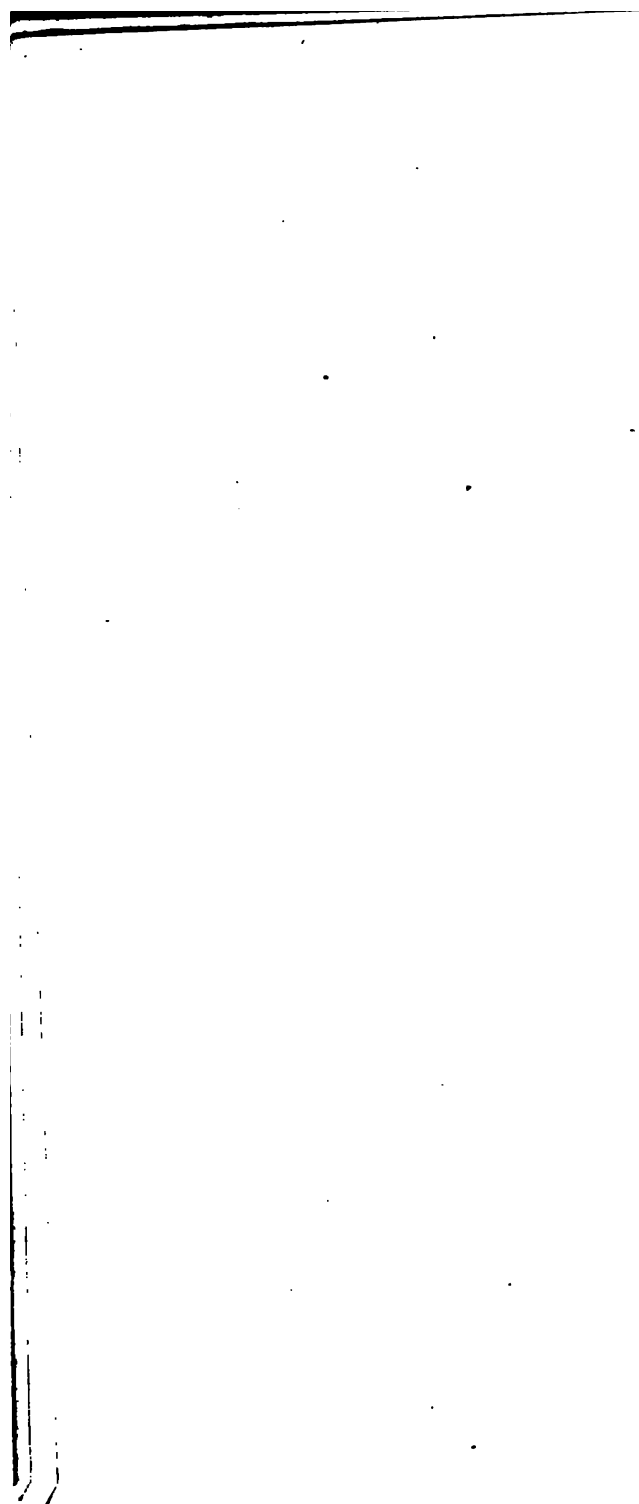
THE following pages have been taken with some slight alterations from the Manuscripts, which have been used of late years in St John's College, and are now printed with the view of saving to the Student the time and trouble, which it has hitherto been necessary to bestow in copying them. The few Propositions of the Seventh and Eighth Sections, now generally read in the University, will be found in the Appendix ; and the Ninth and Eleventh Sections will be published, it is expected, in the course of a few weeks.

St John's College, *Jan.* 1834.

SECTION I.

DEF. THE *Limit* of a continually increasing or decreasing quantity or ratio is that quantity or ratio, to which it continually approximates, but to which, though it may approach nearer than by any assignable difference, it never becomes actually equal.

OBS. The limit of a varying quantity or ratio is frequently called the *ultimate* value of that quantity or ratio; when we say that one quantity is *ultimately* equal to another, it is not to be inferred that the two quantities are ever equal, though their difference may be less than any assignable quantity.

LEMMA I.

Quantities and the ratios of quantities, which tend continually to equality, and whose difference may be made to bear to either of them a ratio less than any finite ratio, have their limits equal.

For if the limits be not equal, let L and $L + D$ represent them; then the difference of the limits to one of them

$$= D : L \text{ or } D : L + D.$$

A

Now since the quantities or ratios tend continually to equality, the ratio of their difference to either of them must always be greater than that of the difference of their limits to either of the limits, that is, than $D : L$ or $D : L + D$, either of which is a finite ratio. But by the hypothesis the ratio of their difference to either of them may be made less than any finite ratio, which is absurd; therefore the limits are not unequal, that is, they are equal.

Cor. Hence if the quantities or ratios be finite, the limit of their difference, as they tend continually to equality, must equal 0. If they be indefinitely great, the limit of their difference may be a finite quantity or ratio, for it would bear to either of them an indefinitely small ratio. Lastly, if they be indefinitely small, it must be a quantity or ratio, which vanishes compared with either, that is, it must be a vanishing quantity or ratio of a higher order than either of them.

Lemma II.

If in any figure AKa, bounded by the straight lines Aa, AK, and the curve line Ka, there be inscribed any number of parallelograms Ab, Bc, Cd,... on equal bases AB, BC, CD,..., and the parallelograms Ba, Cb, Dc.... be completed; then if the number of these parallelograms be increased and their breadths diminished indefinitely, the limit of the sum of each series will be the curvilinear area AKa.

For as their bases are diminished, each series of parallelograms continually approximates to the area AKa. Also the difference between the two series is the sum of the parallelograms ab, bc, cd... which sum is equal to the parallelogram aB, for the base of each is equal to AB, and the sum of their altitudes to that of aB, and by diminishing the bases this difference, and therefore, *a fortiori*,
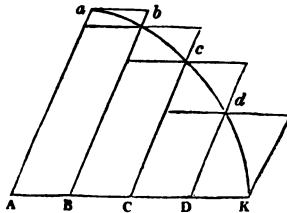
the difference of either series and the area AKa may be made less than any assignable quantity, and therefore, by Lemma 1, the limit of either series is the curvilinear area AKa.

Lemma III.

If the two series of parallelograms be described in the same manner as in the last Lemma, except that their bases are not all equal, the limit of each series, when their bases are diminished indefinitely, is in this case also the curvilinear area AKa.

For take AF equal to the great-est base, and complete the parallel-ogram Fa; then this parallelogram, which is evidently greater than the difference between each series of parallelograms, may, by diminishing the base, be made less than any assignable quantity. Hence the dif-ference between the two series, and therefore, *a fortiori*, the difference between each series and the area AKa, may be made less than any assignable quantity; and they tend continually to equality, therefore, by Lemma 1, the limit of each series is the curvilinear area AKa.
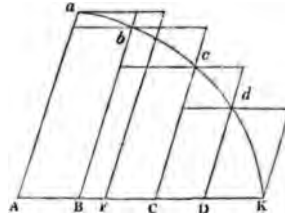
Cor. 1. If the chords ab, bc, $cd...$ be drawn, the limit of the area bounded by Aa, AK and the chords, when the bases AB, BC, $CD....$ are diminished indefinitely, is the curvilinear area AKa, for it always lies between this area, and the inner series of parallelograms.

Cor. 2. The limit of the figure bounded by Aa, AK and the tangents through a, b, $c...$ is the same curvilinear area, since it lies always between the curvilinear area, and the outer series of parallelograms.

Cor. 3. The curve line aK is the limit of the boundary formed by the chords.

Lemma IV.

If in two curvilinear figures there can be inscribed the same number of parallelograms, which, when their number is increased and their breadths diminished indefinitely, are ultimately to each other in a given ratio, the areas of the curvilinear figures will be in that ratio.

Let PQR, pqr be the figures, and let the parallelograms $A_1, A_2, A_3, \dots$ be inscribed in the one, and $a_1, a_2, a_3, \dots$ in the other,

and let $\dfrac{A_1}{a_1} = m + x_1,$ $\quad \dfrac{A_2}{a_2} = m + x_2,$ $\quad \dfrac{A_3}{a_3} = m + x_3,$ $\quad$ &c. = &c.

$x_1\ x_2\ x_3 \dots$ being quantities, which vanish, when the breadths of the parallelograms are diminished indefinitely, so that according to the hypothesis,
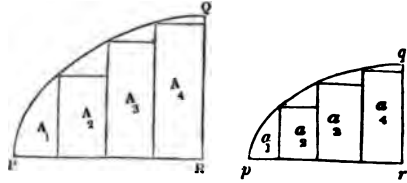
$$\text{limit } \frac{A_1}{a_1} = m = \text{limit } \frac{A_2}{a_2} = \text{limit } \frac{A_3}{a_3} = \&\text{c.}$$

Hence $A_1 = m a_1 + x_1 a_1,$ $A_2 = m a_2 + x_2 a_2,$ $A_3 = m a_3 + x_3 a_3,$ &c. = &c.

$\therefore A_1 + A_2 + A_3 + \dots = m(a_1 + a_2 + a_3 + \dots) + x_1 a_1 + x_2 a_2 + x_3 a_3 + \dots$

$$\therefore \frac{A_1 + A_2 + A_3 + \dots}{a_1 + a_2 + a_3 + \dots} = m + \frac{x_1 a_1 + x_2 a_2 + x_3 a_3 + \dots}{a_1 + a_2 + a_3 + \dots}$$

$$\therefore \text{ limit } \frac{A_1 + A_2 + A_3 + \dots}{a_1 + a_2 + a_3 + \dots} = m + \text{limit } \frac{x_1 a_1 + x_2 a_2 + x_3 a_3 + \dots}{a_1 + a_2 + a_3 + \dots}$$

Now since x_1 vanishes in the limit, $x_1 a_1$ is a vanishing quantity of a higher order than a_1; similarly $x_2 a_2$ vanishes compared with a_2, $x_3 a_3$ compared with a_3, and so on; the number of terms also in the two series is the same, therefore ultimately $x_1 a_1 + x_2 a_2 + x_3 a_3 + \dots$ vanishes compared with $a_1 + a_2 + a_3 + \dots$,

$$\text{or limit } \frac{x_1 a_1 + x_2 a_2 + x_3 a_3 + \dots}{a_1 + a_2 + a_3 + \dots} = 0.$$

$$\text{Also}\ \ \text{limit}\ \ \frac{A_1 + A_2 + A_3 + \ldots}{a_1 + a_2 + a_3 + \ldots} = \frac{\text{area } PQR}{\text{area } pqr},$$

$$\therefore\ \ \frac{\text{area } PQR}{\text{area } pqr} = m.$$

COR. If there be two quantities of any kind, which are divided into the same number of parts, if these parts, when their number is continually increased and the magnitude of each continually diminished, are to each other in a given ratio, the whole quantities will be in that ratio.

For if the parts be substituted for the parallelograms, and the whole quantities for the figures *PQR*, *pqr*, the reasoning will be the same in the two cases.

DEF. 1. A *curve* is a line traced out by a moving point, which is continually changing the direction of its motion.

DEF. 2. One curvilinear figure is said to be similar to another, when any rectilinear figure being inscribed in the first, a similar rectilinear figure may be inscribed in the other.

OBS. The curves and curvilinear figures, treated of in this Section, are always supposed to lie in one plane.

LEMMA V.

The homologous sides of all similar curvilinear figures are proportionals, and their areas are in the duplicate ratio of the sides.

Let *ACB*, *acb* be two similar figures, of which the sides *AB*, *AC*, *BC*, are homologous to *ab*, *ac*, *bc*, respectively; then by definition, if *ADEBC* be a polygon inscribed in *ABC*, a similar polygon *adebc* may be inscribed in *abc*. Join *CD*, *CE*, and *cd*, *ce*, &c., dividing the polygons into the same number of similar triangles,
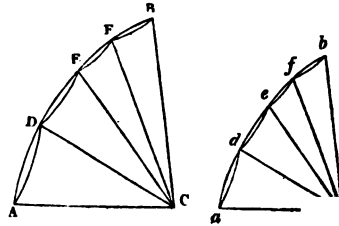
$$\therefore\ AD\ :\ AC = ad\ :\ ac,$$

$$\text{alt}^{do}\ AD\ :\ ad = AC\ :\ ac,$$

$$\text{Similarly}\ DE\ :\ de = DC\ :\ dc = AC\ :\ ac,$$

$$EF\ :\ ef = AC\ :\ ac,$$

$$\dots\dots\dots\dots\dots\dots\dots$$

therefore, componendo

$$AD + DE + EF + \&c.\ :\ ad + de + ef + \dots = AC\ :\ ac.$$

Now this being always true, will be true when the number of sides is increased, and their magnitudes diminished, without limit;

$$\therefore \text{ limit } AD + DE + EF + \dots\ :\ \text{limit } ad + de + ef + \dots = AC\ :\ ac,$$

and therefore by Lem. III, Cor. 3.

$$ADB\ :\ adb = AC\ :\ ac$$

$$= BC\ :\ bc.$$

Again, polygon $ADEBC$: polygon $adebc = AC^2\ :\ ac^2$,

and this being always true will be true in the limit;

$$\therefore \text{ limit polygon } ADEBC\ :\ \text{limit } adebc = AC^2\ :\ ac^2\ ;$$

therefore by Lem. III, Cor. 1,

curvilinear figure ABC : curvilinear fig. $abc = AC^2\ :\ ac^2$

$$= \overline{ADB}|^2\ :\ \overline{adb}|^2$$

$$= BC^2\ :\ bc^2.$$

Cor. If ACB, acb be two similar figures, and CE, ce be equally inclined to AC, ac, then $AC : CE = ac : ce$. Hence also this definition,

Two curves are said to be similar, when there can be drawn in them two distances from two points similarly situated, such, that if any two other distances be drawn equally inclined to the former, the four are proportional.

PROB. Let the chord AB of the curve ACB be produced to b, to describe on Ab a curve similar to ACB.

In ACB take any point P, join AP, and produce AP to p, so that $Ap : Ab = AP : AB$; then if the curve Apb be the locus of all points, whose position is determined in the same manner as that of p, it will be similar to the curve APB.
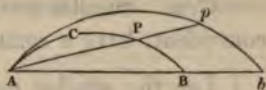
DEF. 1. The *tangent* to a curve AB at A is the straight line, in which the generating point would move, if instead of changing the direction of its motion, it moved on in the direction, which it had at A.

DEF. 2. The curvature of a curve is said to be *continued*, when the curve is wholly convex or concave to a given straight line on the same side of it, and when the change of direction is not abrupt, but gradual; that is, if ATU, BT, (Fig. Lem. VI.) be tangents at A and B, in a curve of continued curvature, the angle BTU as B moves up to A, diminishes through every change of magnitude from its original value and ultimately vanishes.

LEMMA VI.

If ACB be an arc of continued curvature, AB the chord, and ATU the tangent at A, the angle BAT between the chord and tangent, as B moves along the curve towards A, and ultimately coincides with that point, continually diminishes and ultimately vanishes.

Let the tangents at A and B meet in the point the angle BTU measures the change in the direction

motion of the generating point which takes
place in passing from B to A, and since
the curvature is continued, this angle, as B
moves towards and ultimately coincides with
A, continually diminishes and ultimately
vanishes, therefore *a fortiori* the interior
angle BAT continually diminishes and ul-
timately vanishes.
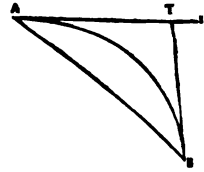
COR. Similar conterminous arcs, which have their chords
coincident, have a common tangent.

Let the similar conterminous arcs APB, apb have their
chords AB, Ab coincident, and let
APp, AQq be any other coincident
chords; then since the curves are si-
milar $AP : Ap = AB : Ab = AQ : Aq$,
therefore the arcs AP, Ap are similar,
that is, the chords of the similar arcs
AP, Ap coincide. Now let P and p
move up to A, the arcs AP, Ap, since they are always similar,
will vanish together, and APp in its ultimate position will be a
tangent to each, that is, the arcs AB, Ab have a common
tangent.
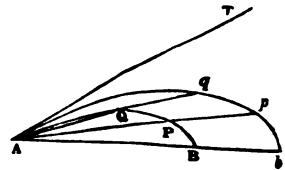
DEF. The *subtense* of an arc is a straight line, drawn from
one extremity of the arc to meet at a finite angle the tangent
to the arc at its other extremity.

LEMMA VII.

*If BD be a subtense of the arc ACB of continued cur-
vature, the chord AB, the arc ACB, and the tangent AD,
when BD moves parallel to itself up to A, are ultimately equal
to each other.*

Produce AD to any fixed
point d, and draw db parallel
to DB to meet AB produced
in b; on Ab describe the arc
Acb similar to ACB, and as
B moves up to A, let Acb so
alter its form as to be always

similar to ACB; hence the two arcs have a common tangent, and the three lines AB, ACB, AD are always proportional to Ab, Acb, Ad. Now as B moves up to A, the angle bAd continually diminishes and ultimately vanishes, (Lemma VI.), the point b moves up to and coincides with d, and therefore Ab and Ad, and therefore, *a fortiori*, the intermediate arc Acb, are ultimately equal. Hence AB, ACB, AD, which are always proportional to them, are ultimately equal to each other.

COR. 1. Since the proof holds whatever be the inclination of BD to the tangent, provided it be finite, if BE be a subtense making any other finite angle with AD, the tangents AE, AD, and the chord and arc are ultimately equal.

COR. 2. Also if the parallelograms $ADBF$, $AEBG$ be completed, since AD, AE are always equal to BF, BG respectively, the lines AD, AE, BF, BG are ultimately equal to the chord and arc; and in all geometrical investigations the ultimate values of all these lines may be used indiscriminately for each other.

LEMMA VIII.

If the straight lines AR, DBR, which meet in R, make with the chord AB, the arc ACB, and the tangent AD, the triangles ABR, $ACBR$, ADR; these three triangles when B moves up to A, are ultimately similar and equal to each other.

Produce AD to a fixed point d, and draw dbr parallel to DBR, meeting AB, AR produced in b, r. On Ab describe the arc Acb similar to ACB, and let it so alter its form, as B moves up to A, as to be always similar to ACB. Then the two arcs will have a common tangent ADd, and the three triangles ABR, $ACBR$, ADR will be always similar to the three Abr, $Acbr$, Adr respectively, and will bear each to each the same ratio, viz. that of $RA^2 : rA^2$; hence, alternando, $ABR : ACBR : ADR = Abr : A$

Now let *BD* move parallel to itself up to *A*, then the angle *bAd* continually diminishes and ultimately vanishes; and *Ab* and therefore the intermediate arc *Acb* ultimately coincide with *Ad*; hence the triangles *Abr*, *Acbr*, are ultimately similar and equal to *Adr*; therefore the triangles *ABR*, *ACBR*, *ADR*, which are always proportional to them, are ultimately similar and equal to each other.

Obs. In the Lemma *RBD* is supposed to move parallel to itself towards *A*, that is, *b* moves along *rd* fixed, and the triangles *Abr*, *Acbr*, *Adr* are always finite; but the same thing will be true, if *RBD* revolve round *R* fixed, in which case also, though *r* moves off to an infinite distance, and the triangles *Abr*, *Acbr*, *Adr* increase indefinitely, they will be ultimately similar and equal to each other.

Lemma IX.

If the right line AE and the arc ABC, given in position, cut each other in a finite angle at A, and the ordinates BD, CE be drawn, making any other given angle with AE; when BD, CE move parallel to themselves up to A, the limiting ratio of area ABD : area ACE equals that of $AD^2 : AE^2$.

Produce *AE* to a fixed point *e*, and take *Ad* in *Ae* such, that $Ad : Ae = AD : AE$. Draw *db*, *ec* parallel to *DB*, or *EC*, meeting *AB*, *AC* produced in *b*, *c*; and on *Ac* describe an arc similar to *ABC* : this arc shall pass through *b*, for by similar triangles and by construction,

$$AB : Ab = AD : Ad = AE : Ae = AC : Ac,$$

and therefore (Cor. Lemma v.) *b* is a point in the arc. As *B* and *C* move up to *A*, let the curve *Abc* so alter its form as to

be always similar to ABC, then the area ABD will be always similar to Abd, and ACE to Ace.   Hence

$$\text{area } ABD \ : \ \text{area } Abd = AD^2 = Ad^2 = AE^2 \ : \ Ae^2$$

$$= \text{area } ACE \ : \ \text{area } Ace,$$

$$\therefore \ \text{area } ABD \ : \ \text{area } ACE = \text{area } Abd \ : \ \text{area } Ace.$$

Also the two arcs being similar have a common tangent at A, let this be $AFGfg$; and let BD, CE move parallel to themselves up to A; then the angle cAg continually diminishes and ultimately vanishes, and therefore

$$\text{L. R.* area } Abd \ : \ \text{area } Ace = \text{L. R. } \triangle Afd : \triangle Age$$

$$= \text{L. R. } Ad^2 \ : \ Ae^2$$

Hence L. R. area ABD : area ACE = L. R. area Abd : area Ace

$$= \text{L. R. } Ad^2 \ : \ Ae^2$$

$$= \text{L. R. } AD^2 \ : \ AE^2.$$

LEMMA X.

The spaces, described from rest by a body acted on by any finite force, are in the beginning of the motion as the squares of the times, in which they are described.

DEF.   A finite accelerating or retarding force is such, that the ratio of the time to the velocity generated or destroyed in that time is finite.

Let the straight line AK represent the time of the body's motion from rest, and Kk, drawn at right angles to AK, the last acquired velocity; suppose the time divided into equal intervals AB, BC, CD &c., and let Bb, Cc, Dd &c., drawn at right angles to AK, represent the velocities acquired in the times AB, AC, AD &c.; let $Abcdk$ be the curve passing through the extremities of all the ordinates thus drawn ; and complete the parallelograms Ab, Bc, Cd &

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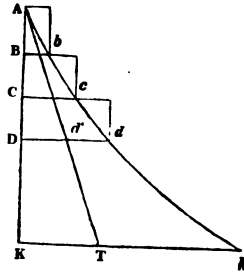
* L. R. signifies "limit of the ratio" or "limiting ratio."

If now the force be supposed to act by impulses, which would cause the body to move uniformly during the times *AB, BC, CD* &c., with the velocities *Bb, Cc, Dd* &c. respectively, the spaces described in the 1st, 2d, 3d &c. intervals will be represented by the parallelograms *Ab, Bc, Cd* &c. On this supposition therefore, the space described in time *AD* : space in time *AK* = sum of the parallelograms in the former case : sum in the latter; and this being true always, will be true when the intervals are diminished and their number increased indefinitely, in which case the force, which was supposed to act by impulses, approximates to a continued force, and the sums of the parallelograms to the areas *ADd, AKk.*

Hence

space in time *AD* : space in time *AK* = area *ADd* : area *AKk.*

Let the tangent at *A* cut *Kk* in *T* ; now, the force being finite, the ratio *AK* : *Kk* is always finite ; ∴ *AK* : *KT*, which equals L. R. *AK* : *Kk* is a finite ratio, and therefore,

$$\tan KAT \left(= \frac{KT}{KA} \right) \text{ is finite,}$$

or *KA* makes a finite angle with the curve at *A* ;

Hence by Lemma IX,

L. R. area *ADd* : area *AKk* = L. R. *AD²* : *AK²*,

and therefore in the beginning of the motion, space ∝ (time)^z.

COR. 1. Force is measured by the velocity generated in any time, divided by the time, the force being supposed to remain constant for that time. Hence if *Dd'* be the velocity generated by the force at *A*, continued constant, in time *AD*,

$$F \text{ at } A = \frac{Dd'}{AD},$$

and this being always true, will be true when AD is diminished indefinitely,

$$\therefore \ F = \text{limit} \ \frac{Dd'}{AD} = \text{limit} \ \frac{Dd}{AD}$$

$$= \frac{KT}{AK} = \frac{KT \cdot AK}{AK^2} = \frac{2 \text{ triangle } AKT}{AK^2} = 2 \text{ limit} \ \frac{\text{area } AKk}{AK^2}$$

$$= 2 \text{ limit} \ \frac{\text{space}}{(\text{time})^2} \ .$$

Cor. 2. The effect produced by F upon the body is independent of any motion which it may have, when F begins to act upon it. Hence generally if S be the space, through which a force F, acting on a body moving in any orbit, draws the body in T''' from the place it would have occupied if the extraneous force had not acted, $F = 2 \text{ limit} \ \dfrac{S}{T^2}$.

On the Curvature of Curve Lines.

Prop. I. If in PR, pr tangents at the points P, p in the curves PQ, pq, PR be taken equal to pr, and the subtenses QR, qr be drawn equally inclined to them, then when QR, qr move parallel to themselves to P, p,

$$\frac{\text{curvature of } PQ \text{ at } P}{\text{curvature of } pq \text{ at } p} = \text{limit} \ \frac{QR}{qr} \ .$$
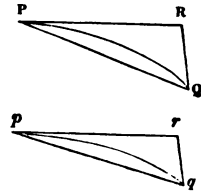
Draw the chords PQ, pq,

$$\text{then} \ \frac{\text{curvature of } PQ \text{ at } P}{\text{curvature of } pq \text{ at } p} = \frac{\text{angle of contact at } P}{\text{angle of contact at } p}$$

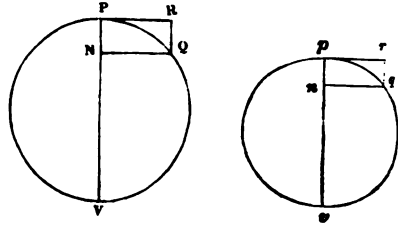$$= \text{limit} \ \frac{\text{angle } QPR}{\text{angle } qpr} \ .$$

$$= \text{limit } \frac{\sin QPR}{\sin qpr}$$

$$= \text{limit } \frac{\dfrac{QR}{RP} \sin R}{\dfrac{qr}{rp} \sin r}$$

$$= \text{limit } \frac{QR}{qr}.$$

PROP. II.   The curvatures in different circles vary inversely as the diameters.

Let PQV, pqv be two circles, draw the diameters PV, pv, and the tangents PR, pr. Take $PR = pr$, and draw the subtenses QR, qr parallel to the diameters, and QN, qn parallel to the tangents;

$$\text{then } \frac{QR}{qr} = \frac{PN}{pn} = \frac{QN^2}{NV} \div \frac{qn^2}{nv} = \frac{nv}{NV},$$

$$\therefore \frac{\text{curvature at } P}{\text{curvature at } p} = \text{limit } \frac{QR}{qr}$$

$$= \text{limit } \frac{nv}{NV}$$

$$= \frac{pv}{PV}$$

$$\text{or the curvature} \propto \frac{1}{\text{diameter}}.$$

Cor. Hence in the same circle the curvature is the same at every point.

From this property of the circle, and also because by varying the diameter it may be made to have any curvature we please, the circle is made use of to measure the curvature at any proposed points of other curves.

Def. The *circle of curvature* at any point of a curve is that circle, which has the same tangent and curvature as the curve has at that point.

Hence if QqR be a common subtense to the curve PQ and the circle Pq, and limit $\dfrac{QR}{qR} = 1$, Pq will be the circle of curvature at P.

The radius, diameter and chord of the circle of curvature are generally called the radius, diameter, and chord of curvature.

Prop. III. If PqV be the circle of curvature at any point P, and PV a chord drawn in any given direction, then

$$PV = \text{limit } \frac{(\text{arc})^2}{\text{subtense parallel to the chord}} .$$
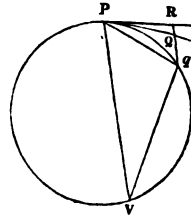
Take PQ a small arc of the curve, through Q draw the subtense RQq parallel to PV, and join Pq, qV; then since the triangles PRq, PqV are evidently similar,

$$PV = \frac{Pq^2}{qR}$$

Now this being true whatever be the magnitude of R, will be true, when RQr moves parallel to itself up to P, in which case $Pq = PQ$ ultimately, and $qR = QR$ ultimately,

$$\therefore \; PV = \text{limit} \; \frac{Pq^2}{qR}$$

$$= \text{limit} \; \frac{(\text{arc } PQ)^2}{QR} \; .$$
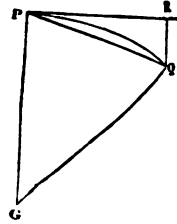
Cor.   Hence the diameter of curvature

$$= \text{limit} \; \frac{(\text{arc})^2}{\text{subtense perpendicular to the tangent}} \; .$$

Prop. IV.   If in the curve PQ, PG and QG, drawn perpendicular to the tangent PR and the chord PQ respectively, intersect in G, then when Q moves up to P, the limit of PG is the diameter of curvature at P.

Draw the perpendicular subtense QR,

Then by similar triangles PQR, PGQ

$$PG = \frac{PQ^2}{QR} ;$$

$$\therefore \; \text{limit } PG = \text{limit} \; \frac{PQ^2}{QR}$$

$$= \text{limit} \; \frac{(\text{arc } PQ)^2}{QR}$$

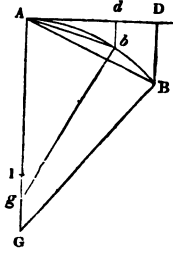$$= \text{diameter of curvature at } P.$$

Def.   The curvature of a curve at any point is said to be finite, when the diameter of curvature at that point is finite.

Lemma XI.

In curves of finite curvature the limiting ratio of the subtenses equals that of the squares of the conterminous arcs.

Let AbB be the curve having a finite curvature at A;

First, Let the subtenses bd, BD be perpendicular to the tangent at A. Draw bg, BG at right angles to the chords Ab, AB, and let them meet AgG, which is drawn at right angles to the tangent AD, in the points g and G.

Then as b and B move up to A, g and G move up to I, the extremity of the diameter of curvature of A, as their limit. (Prop. iv.)

Now by similar triangles,

$$BD = \frac{AB^2}{AG}, \quad bd = \frac{Ab^2}{Ag},$$

$$\therefore BD : bd = \frac{AB^2}{AG} : \frac{Ab^2}{Ag},$$

$$\therefore \text{L. R. } BD : bd = \text{L. R. } \frac{AB^2}{AG} : \frac{Ab^2}{Ag}$$

$$= \text{L. R. } AB^2 : Ab^2,$$

(since AG, Ag are ultimately equal to AI)

$$= \text{L. R. } (\text{arc } AB)^2 : (\text{arc } Ab)^2.$$

Secondly, Let the subtenses be inclined at any equal angles to the tangent. Draw BE, be perpendicular to the tangent: then by similar triangles,

C
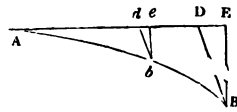
$$BD : BE = bd : be,$$

alternando $BD : bd = BE : be$;

$$\therefore \text{ L. R. } BD : bd = \text{L. R. } BE : be$$

$$= \text{L. R. (arc } AB)^2 : (\text{arc } Ab)^2.$$

Thirdly, Let the subtenses, inclined at unequal angles to the tangent, converge to a point, and revolve round that point fixed, or approach to A according to any other given law.

Let O be the point in which DB, db meet when produced; draw BE, be always parallel to AO; then since the angles at D and d are always finite, AO must always be finite, and L. R. $DO : AO$ will be a ratio of equality, as also L. R. $do : AO$.
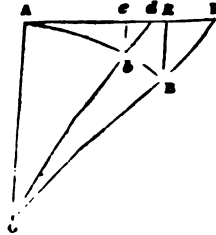
But $BD : BE = DO : AO$

and $bd : be = do : AO$ }, always and therefore ultimately;

$$\therefore \text{ L. R. } BD : BE = \text{L. R. } bd : be;$$

$$\therefore \text{ L. R. } BD : bd = \text{L. R. } BE : be$$

$$= \text{L. R. (arc } AB)^2 : (\text{arc } Ab)^2.$$

Cor. 1. Hence by Lemma vii. the limiting ratio of the subtenses will equal that of the squares of the arcs, chords, and tangents.

Theorem. *The subtense of an arc is ultimately equal to four times the parallel sagitta.*

Def. The sagitta of an arc is a line drawn at a finite angle to the chord from its middle point to meet the arc.

Let *BD* be a subtense of the arc *AB*, *EC* the sagitta parallel to it, bisecting the chord in *E*, and produced to meet the tangent in *F*.
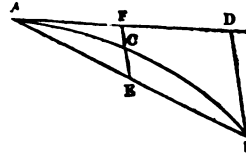
Then by similar triangles,

$$AF = \tfrac{1}{2}AD, \text{ and } EF = \tfrac{1}{2}BD.$$
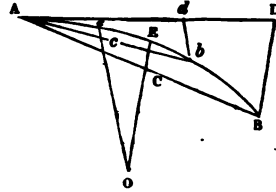
Also by the Lemma,

$$\text{L. R. } CF : BD = \text{L. R. } AF^2 : AD^2$$

$$= 1 : 4$$

$$\therefore \text{L. R. } CE : BD = 1 : 4.$$

COR. 2.   The limiting ratio of the sagittæ, which bisect the chords and converge to a given point, equals that of the squares of the arcs, chords, and tangents.

Let *EC*, *ec* be the sagittæ of the arcs *AEB*, *Aeb*, bisecting the chords *AB*, *Ab* in *C*, *c*; draw the subtenses *BD*, *bd* respectively parallel to them;

then $\text{L. R. } EC : BD = 1 : 4$

$$= \text{L. R. } ec : bd;$$

$$\therefore \text{L. R. } EC : ec = \text{L. R. } BD : bd;$$

$$= \text{L. R. (arc } AB)^2 : (\text{arc } Ab)^2$$

$$= \text{L. R. (chord } AB)^2 : (\text{chord } Ab)^2$$

$$= \text{L. R. (tangent } AD)^2 : (\text{tangent } Ad)^2.$$

COR. 3.   Hence if a body describe the arcs *AB*, *Ab* with any given velocity, the limiting ratio of the sagittæ will be that of the squares of the times, in which they are described.

Cor. 4. If the subtenses DB, db be perpendicular to the tangent, as in the first case of the Lemma,

$$\triangle ADB \,:\, \triangle Adb = AD \,.\, DB \,:\, Ad \,.\, db;$$

$$\therefore \text{ L. R. } \triangle ADB \,:\, \triangle Adb \quad = \text{L. R. } AD \,.\, DB \,:\, Ad \,.\, db$$

$$= \text{L. R. } AD^2 \,:\, A d^2$$

$$\text{or } = \text{L. R. } DB^{\frac{3}{2}} \,:\, db^{\frac{3}{2}}.$$

Cor. 5. Since L. R. $DB : db = $ L. R. $AD^2 : Ad^2$, the limiting form to which every curve of finite curvature approximates is the common parabola.

Hence L. R. area ADB : area $Adb = $ L. R. $\frac{2}{3} AD \,.\, DB \,:\, \frac{2}{3} Ad \,.\, db$

$$= \text{L. R. } AD^3 \,:\, A d^3$$

$$\text{or } = \text{L. R. } DB^{\frac{3}{2}} \,:\, db^{\frac{3}{2}}.$$

Scholium to Lemma XI.

It was proved in the Lemma that if the curvature be finite, the subtense varies ultimately as the square of the conterminous arc; conversely,

If the subtense vary ultimately as the square of the arc, the curvature is finite, and if it vary according to any other power of the arc, the curvature is infinitely great or infinitely small.

Let PQ and Pq be arcs of a curve and circle, having a common tangent PR, and let RQq be a common subtense.

Since in the circle $qR \propto$ ult. PR^2, let $qR = a \,.\, PR^2$ ultimately,

and suppose that $QR \propto$ ult. PR^n and $= b \,.\, PR^n$ ultimately

$$\therefore \frac{\text{curvature of } PQ}{\text{curvature of } Pq} = \text{limit } \frac{QR}{qR} = \frac{b}{a} \,.\, \text{limit } PR^{n-2}.$$

If $n = 2$, the curvature of the curve PQ bears a finite ratio to that of the circle, and is therefore finite. If n be greater than 2, limit $PR^{n-2} = 0$, and therefore the curvature of PQ is infinitely small compared with that of Pq, and the curve will lie between Pq and the tangent. If n be less than 2, limit $PR^{n-2} = \infty$, and therefore the curvature of PQ is infinitely great, and the curve will lie below Pq.

Cor. Since an infinite number of values may be given to n, to each of which there will be a corresponding curve, an infinite number of curves may be described between Pq and the tangent, corresponding to values of n greater than 2, and an infinite number below Pq, corresponding to values of n less than 2.
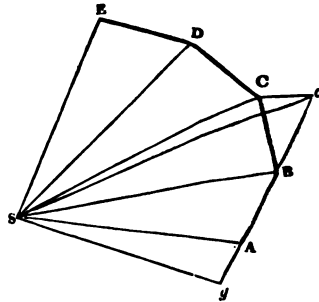
SECTION II.

On the motion of a body, considered as a point, moving in a non-resisting medium, and attracted to a single fixed center of force.

PROP. I. *If a body move in any orbit about a fixed center of force, the areas, described by lines drawn from the center to the body, lie in one plane, and are proportional to the times of describing them.*

Let S be the center of force; and suppose a body unattracted by the force in S to describe the straight line AB with a uniform velocity in a given time (T). Then if suffered to proceed, it would move on uniformly in the direction of AB produced, and describe $Bc = AB$ in the next interval (T); but at B suppose an instantaneous impulse communicated to it in direction BS, which causes it to move in direction BC; draw cC parallel to BS, then by the principles of Mechanics, the body at the end of the second interval will be found at C. Join SA, SB, Sc, SC. Since cC is parallel to BS, the triangle $SBC = SBc = SAB$, since $Bc = AB$; and these triangles are in the same plane, as no force has acted to draw the body out of the plane SAB. Similarly, if impulses be communicated at the end of every interval of T'', in directions tending always

to S, causing the body to describe CD, DE, &c. in the third, fourth, &c. intervals, the triangles SAB, SBC, SCD, &c. will be all equal, and will lie in the same plane; and their bases AB, BC, CD, &c. are described in equal times, therefore the area of any number of these triangles, or the polygon $SABCDE$ varies as the time of describing it. Now let the number of intervals be increased, and the magnitude of each diminished indefinitely, then the polygon approximates to a curvilinear area, and the sum of the impulses to a continued force always tending to S, as their limits; and what was proved of those quantities is true of their limits, and therefore the curvilinear area described in any time is proportional to the time.

OBS. The area, described by the line joining S and the body, is frequently called the area described by the body round S.

COR. 1. If V be the velocity of the body at A, and p the perpendicular from S upon the tangent at that point, the area described in $t'' = \frac{1}{2} p \cdot t \cdot v$.

Draw Sy perpendicular to AB; then since AB is ultimately the tangent at A, limit of $Sy = p$. Also if t be divided into n equal intervals, and AB be the space described in the first interval, the force in S being supposed, as in the Prop., not to act, $AB = \dfrac{t}{n} \cdot v$.

Hence, polygonal area described in $t'' = n \cdot$ triangle SAB

$$= n \cdot \frac{1}{2} \cdot Sy \cdot \frac{t}{n} \cdot v$$

$$= \frac{1}{2} Sy \cdot t \cdot v;$$

and the same is true in the limit,

$$\therefore \text{ curvilinear area described in } t'' = \frac{1}{2} p \cdot t \cdot v.$$

COR. 2. Hence the time of describing any part of the orbit

$$= \frac{2}{p \cdot v} \cdot \text{ area described.}$$

Cor. 3.   If $t = 1$, area described in $1'' = \frac{1}{2}\,p\,.\,v.$

Hence in different orbits, the velocity at any point

$$\propto \frac{\text{area described in } 1''}{\text{perpendicular from } S \text{ upon the tangent}},$$

and in the same orbit, the velocity

$$\propto \frac{1}{\text{perpendicular upon the tangent}}.$$

Prop. II.   *If a body, moving in a curve, describe in one plane areas proportional to the times by lines drawn from the body to any point, the body is acted on by centripetal forces all tending to that point.*   (Vide Fig. Prop. I.)

Let S be the point, about which areas proportional to the times are described; and suppose as in Prop. 1. that a body, unattracted by the force in S, describes the straight line AB in a given time T.

In AB produced take $Bc = AB$; then if suffered to proceed, the body would be at c at end of the second interval of T''.   But at B suppose an impulse communicated, which causes it to describe BC in the second interval, such that the triangle $SBC = SAB$.   Join cC, Sc.

Then the triangle $SBC = SAB = SBc$, therefore cC is parallel to BS, and therefore by the principles of Mechanics the impulse communicated at B tends to S.   Similarly if D, E, &c. be the places of the body at the ends of the third, fourth, &c. intervals of T'', so that the triangles SAB, SBC, SCD, &c. are all equal, all the impulses communicated may be shewn to tend to S.

Now suppose the number of intervals increased, and the magnitude of each diminished indefinitely, then the limit of the

polygon is the curvilinear area and that of the sum of the impulses a continued force tending to S; and the above reasoning still holds in the limit, therefore the body is acted on by a continued force tending to S.

PROP. III. COR. Draw CV parallel to AB meeting SB in V and join AV. Then $CV = Bc = AB$, $\therefore$ AV is equal and parallel to CB, or $ABCV$ is a parallelogram. Draw the diagonal CA bisecting BV in m.

Now suppose $S'A'B'C'D'$ to be another orbit, in which the chords $A'B'$, $B'C'$ are described in the same time as either of the chords AB or BC; and let the same construction be made as in the former orbit, then impulse at B : impulse at $B' = c\,C$: $c'C' = Bm$: $B'm'$ and therefore force at B : force at $B' = $ L. R. Bm : $B'm'$; or the centripetal forces in different orbits are in the limiting ratio of the sagittæ of arcs described in equal times, which ultimately pass through the centers of force.

PROP. IV. *The centripetal forces, by which bodies describe different circles with uniform velocities, tend to the centers of the circles, and are as the squares of the arcs, described in the same time, divided by the radii.*

Since in each circle the motion is uniform, the arcs described are proportional to the times. But the sectors, i. e. the areas described, are as the arcs on which they stand; and are therefore proportional to the times. Hence (Prop. II.) the forces tend to the centers of the circles.

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Again let CAB, cab be arcs described in the same time in the circles, whose centers are S, s, and let A, a be their middle points; join AB, ab, and draw the diameters ASV, asv cutting the chords CB, cb in D, d; then (Prop. II. Cor.)

Force at A : force at a = L. R. AD : ad

$$= \text{L. R. } \frac{(\text{chord } AB)^2}{AV} : \frac{(\text{chord } ab)^2}{av}$$

$$= \text{L. R. } \frac{(\text{arc } AB)^2}{AS} : \frac{(\text{arc } ab)^2}{as}.$$

Take AE, ae any other arcs described in equal times;

then AE : $ae = AB$: ab,

and this being true whatever be the magnitudes of AB, ab will be true when they are diminished indefinitely,

$$\therefore AE : ae = \text{L. R. } AB : ab,$$

and therefore force at A : force at $a = \dfrac{AE^2}{AS} : \dfrac{ae^2}{as}$.

Cor. 1. Since AE = velocity × time, if V = velocity of the body, R = radius of the circle, and the time be given,

$$F \propto \frac{V^2}{R}.$$

Cor. 2. Let P equal the periodic time, then since $s = tv$,

$$2\pi R = P . V; \quad \therefore F \propto \frac{R^2}{P^2 . R} \propto \frac{R}{P^2}.$$

Cor. 3. If P be given, $F \propto R$. If $P \propto R^{\frac{3}{2}}$, $F \propto \dfrac{1}{R^2}$;

and generally if $P \propto R^n$, $F \propto \dfrac{1}{R^{2n-1}}$.

PROP. V. *Given the velocities of a body, and the directions of its motion at three points of its orbit, to determine the position of the center of force.*

Let *PM*, *MQN*, *NR* be the directions, in which the body is moving at the three points *P*, *Q*, *R*; draw *Pp*, *Qq*, *Rr* at right angles to these lines respectively, and such that

$$Pp \; : \; Qq = \text{velocity at } Q \; : \; \text{velocity at } P,$$

$$\text{and } Qq \; : \; Rr = \text{velocity at } R \; : \; \text{velocity at } Q.$$

Through *p*, *q*, *r* draw *pm*, *mn*, *nr* respectively parallel to *PM*, *MN*, *NR*; join *Mm*, *Nn* and produce them to meet in *S*; *S* will be the center of force.
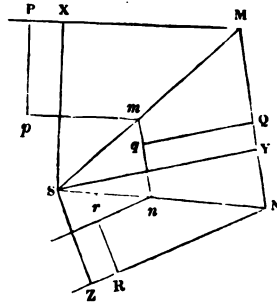
Draw *SX*, *SY*, *SZ* perpendicular to *PM*, *MN*, *NR*, respectively,

$$\text{then } \frac{SX}{SM} = \frac{Pp}{Mm},$$

$$\text{and } \frac{SM}{SY} = \frac{Mm}{Qq};$$

$$\therefore \; \frac{SX}{SY} = \frac{Pp}{Qq} = \frac{\text{velocity at } Q}{\text{velocity at } P}.$$

$$\text{Similarly } \frac{SY}{SZ} = \frac{\text{velocity at } R}{\text{velocity at } Q}.$$

Hence the perpendiculars, drawn from *S* upon the tangents at *P*, *Q*, *R*, are inversely as the velocities at those points; therefore *S* must be the center of force. (Prop. 1. Cor. 3.)

Prop. VI. *A body moving round a fixed center of force*
S, describes the arc PQ in T''; if F be the central force at P,
and QR a subtense parallel to SP, when PQ and T are di-
minished indefinitely,

$$F = 2 \operatorname{limit} \frac{QR}{T^2}.$$

The motion of the body on leaving P
is compounded of two motions, one uniform
in direction of the tangent PR, the other
variable, arising from the action of S and
taking place in direction of the line join-
ing the body with S; therefore since RQ
is parallel to PS, its ultimate value, when
T is diminished indefinitely, will be the
space described by the action of S in that
time.
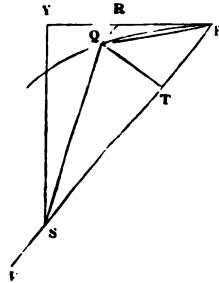
Hence (Lemma x. Cor. 2.) $F = 2 \operatorname{limit} \dfrac{QR}{T^2}.$ *

Cor. 1. Draw QT perpendicular to SP, and join SQ,
QP; let $h = 2$ area described in $1''$,

$$\text{then } \frac{T''}{1''} = \frac{2 \text{ area } PSQ}{h},$$

also limit area PSQ = limit triangle PSQ

$$= \tfrac{1}{2} \operatorname{limit} QT \cdot SP;$$

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* The above expression for the force being obtained independently of the preceding
propositions, it is not necessary that the areas described should be proportional to the
times. It is therefore true in orbits described round several centers of force, in which
case the expression represents the magnitude of the resultant of all the forces acting on
the body at the point P. It is clear, however, that the equable description of areas is
supposed to be preserved in the three succeeding corollaries. The result in Cor. 4. is
general, and might easily be obtained from Cor. 3, in the particular case of the areas
being described equally, by substituting for h its value $Sy \cdot V$, obtained in Cor. 2.
Prop. 1.

$$\therefore \ F = 2 \ \text{limit} \ \frac{QR}{T^2}$$

$$= 2 \frac{h^2}{4} \cdot \text{limit} \ \frac{QR}{\frac{1}{4} \, QT^2 . SP^2}$$

$$= \frac{2h^2}{SP^2} \cdot \text{limit} \ \frac{QR}{QT^4} \cdot$$

COR. 2.   Draw Sy perpendicular to the tangent PR, then since the angle QPR ultimately vanishes, the triangles QPT, SPy are ultimately similar;

$$\therefore \ \text{limit} \ \frac{QT}{PQ} = \frac{Sy}{SP},$$

$$\therefore \ \text{limit} \ \frac{QR}{QT^2} = \frac{SP^2}{Sy^2} \, \text{limit} \ \frac{QR}{PQ^2},$$

$$\therefore \ F = \frac{2h^2}{Sy^2} \, \text{limit} \ \frac{QR}{PQ^2} \cdot$$

COR. 3.   If PV be the chords of curvature at P through S,

$$PV = \text{limit} \frac{PQ^2}{QR}, \quad \therefore \ F = \frac{2h^2}{Sy^2 . PV} \cdot$$

OBS.   If $A =$ the area described in P'', $h = \dfrac{2A}{P}$, which value may be substituted for h in the above expressions for the force.

COR. 4.   *The space, through which a body must descend from rest by the action of the force at P continued constant, in order to acquire the velocity at P, is $\frac{1}{4}$th of the chord of curvature PV.*

Since limit $\dfrac{PR}{PQ} = 1, \quad F = 2 \, \text{limit} \ \dfrac{QR}{T^2} = 2 \, \text{limit} \ \dfrac{QR}{PQ^2} \cdot \left(\dfrac{PR}{T} \right)^2 \cdot$

Now limit $\dfrac{QR}{PQ^2} = \dfrac{1}{PV}$, and limit $\dfrac{PR}{T}$ = velocity at $P = V$;

$$\therefore \ F = \dfrac{2V^2}{PV} \ \text{ and therefore, } \ V^2 = F \cdot \dfrac{PV}{2}.$$

Let S = space due to V by the action of F continued constant,

$$\text{then, } \ V^2 = 2FS,$$

hence equating this to the above expression for V^2, we have

$$S = \tfrac{1}{4} PV.$$

Cor. 5.   To find the velocity and periodic time of a body, revolving in a circle and acted on by a centripetal force tending to the center of the circle.

Here $PV =$ the diameter $= 2R$, $\therefore \ v = \sqrt{F \cdot R}$;

$$\text{Also } P = \dfrac{\text{circumference}}{\text{velocity}} = \dfrac{2\pi R}{\sqrt{F \cdot R}} = 2\pi \sqrt{\dfrac{R}{F}}.$$

Cor. 6.   If V, v be the velocities at P, p, points similarly situated in similar orbits, described round S, s centers of force, also similarly situated,

$$\text{Force at } P \ (F) \ : \text{force at } p \ (f) \ = \dfrac{V^2}{SP} : \dfrac{v^2}{sp}.$$

Let PQ, pq be arcs described in equal times, QR, qr subtenses parallel to SP, sp, and PV, pv chords of curvature at P, p through S, s.

Then since the times are equal,
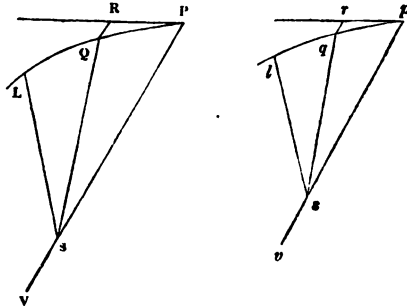
$$F : f = \text{L. R. } QR : qr$$

$$= \text{L. R. } \frac{PQ^2}{PV} : \frac{pq^2}{pr},$$

$$\text{also } V : v = \text{L. R. } \frac{PQ}{T} : \frac{pq}{T}$$

$$= \text{L. R } PQ : pq,$$

and since P and p are points similarly situated in similar orbits,

$$SP : sp = PV : pv,$$

$$\therefore F : f = \frac{V^2}{SP} : \frac{v^2}{sp}.$$

Cor. 7.   If similar arcs of similar orbits be described in times T, t round S, s, centers of force similarly situated, (Fig. Cor. 6.)

$$F : f = \frac{SP}{T^2} : \frac{sp}{t^2}.$$

Let PL, pl be similar arcs described in times T, t, and take PQ, pq other similar arcs described in times P, p; QR, qr subtenses parallel to SP, sp; then

$$F : f = \text{L. R. } \frac{QR}{P^2} : \frac{qr}{p^2},$$

join SQ, SL, sq, si.

Then $T : P = \text{area } PSL : \text{area } PSQ$

$$= \text{area } psl : \text{area } psq, \text{ by similar figures}$$

$$= t : p,$$

$$\therefore T : t = P : p;$$

and this, being always true, will be true when P and p are diminished indefinitely,

$$\therefore \ T \ : \ t = \text{L. R. } P \ : \ p,$$

and by similar figures,

$$SP \ : \ sp = QR \ : \ qr \text{ always and therefore in}$$

the limit;

$$\therefore \ F \ : \ f = \frac{SP}{T^2} \ : \ \frac{sp}{t^2}.$$

PROP. VII. *A body revolves in the circumference of a circle, to find the law of force by which it is attracted to a given point.*

Let PAV be the circumference of the circle, and S the center of force; PQ a small arc, QR a subtense parallel to SP, QT perpendicular to SP. Let RQ, and PS produced if necessary, meet the circumference in G, V; draw the diameter PI, join IV, and produce TQ, PR to meet in Z. The triangles PTZ, PVI are evidently similar.
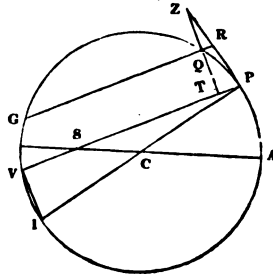
Hence $\dfrac{QR \cdot RG}{QT^2} = \dfrac{RP^2}{QT^2}$ (Euc. III. 36.) $= \dfrac{ZP^2}{ZT^2} = \dfrac{PI^2}{PV^2}.$

Now let Q move up to P,

$$\text{then limit } \frac{QR}{QT^2} = \text{limit } \frac{PI^2}{PV^2 \cdot RG}$$

$$= \text{limit } \frac{PI^2}{PV^3}, \text{ since limit } RG = PV.$$

$$\therefore \ F = \frac{2h^2}{SP^2} \cdot \text{limit} \ \frac{QR}{QT^2}$$

$$= \frac{2h^2}{SP^2} \cdot \frac{PI^2}{PV^3} = \frac{8h^2 R^2}{SP^2 \cdot PV^3},$$

if R = radius of the circle.

Let μ represent that part of the expression for F, which in the same orbit is invariable; then in this case,

$$\mu = 8h^2 R^2,$$

Hence $\quad F = \dfrac{\mu}{SP^2 \cdot PV^3}$,

and therefore in the same circle $\propto \dfrac{1}{SP^2 \cdot PV^3}$.

Cor. 1. To find the velocity at any point.

$$V^2 = F \cdot \frac{PV}{2} = \frac{4h^2 \cdot R^2}{SP^2 \cdot PV^2}, \quad \text{or} = \frac{\mu}{2 \cdot SP^2 \cdot PV^2};$$

$$\therefore \ v = \frac{2hR}{SP \cdot PV}, \quad \text{or} = \sqrt{\frac{\mu}{2}} \cdot \frac{1}{SP \cdot PV}.$$

Obs. The quantity (μ) here introduced is that part of the general expression for the centripetal force in any orbit, which is invariable for all points in that orbit, and may always be determined, if the actual force at any given point be known. The force, by which a body is retained in a given curve, is in most cases undergoing a continual change in magnitude; but its magnitude at any given point is to be estimated by the effect it would produce, that is, by the velocity it would generate in a unit of time from rest, supposing it to remain constant for that time. Hence if a second and a foot be the units of time and space, the magnitude of the centripetal force at any

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point is represented by twice the number of feet, which it would cause a body to describe from rest in $1''$; if for instance, it draws a body from rest through 10 feet in $1''$, its magnitude will be 20, and it will be to the force of gravity in the ratio of $20 : 32\frac{1}{4}$ or of $100 : 161$. Suppose then in the preceding proposition, that the force at A, the extremity of the diameter through S, would if continued constant draw a body through (f) feet in $1''$;

$$\text{then } \frac{\mu}{SA^2 \cdot (2R)^3} = 2f;$$

$$\therefore \mu = 2f \cdot SA^2, (2R)^3.$$

COR. 2.   Let S be in the circumference, then $PV = SP$.

Hence $F = \dfrac{8h^2R^2}{SP^5}$,   or $= \dfrac{\mu}{SP^5}$;   and therefore, $\propto \dfrac{1}{SP^5}$,

$$V = \frac{2hR}{SP^2}, \quad \text{or} = \sqrt{\frac{\mu}{2}} \cdot \frac{1}{SP^2}.$$

COR. 3.   To compare the forces, by which a body, attracted separately to two centers of force, may describe the same circle in the same periodic time.

Let R and S be the two centers of force; produce PR, PS if necessary to meet the circumference in U, V; draw SG parallel to RP to meet the tangent at P in G, and join UV; then the triangles SPG, PVU are evidently similar,
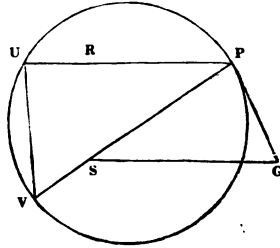
$$\therefore \frac{SG}{SP} = \frac{PV}{PU}, \quad \text{or } SG = \frac{PV \cdot SP}{PU}.$$

Also since the periodic time is the same, h, which $=$ $\dfrac{2 \text{ area of circle}}{\text{periodic time}}$, is the same for both centers, hence

$$F \text{ to } R \; : \; F \text{ to } S = \frac{1}{RP^2 . PU^3} \; : \; \frac{1}{SP^2 . PV^3}$$

$$= \frac{PV^3 . SP^3}{PU^3} \; : \; RP^2 . SP$$

$$= SG^3 \; : \; RP^2 . SP.$$

Cor. 4. What has been proved in the last corollary in the case of the circle is true of any orbit described round two centers of force separately in the same periodic time. For if PUV be the circle of curvature at P, the expression for F, viz. $2 \text{ limit } \dfrac{QR}{QT^2}$, is the same in the curve and circle, and therefore what has been proved in the one case is true in the other. Hence generally in any orbit described in the same time round two centers of force,

$$F \text{ to } R \; : \; F \text{ to } S = SG^3 \; : \; RP^2 . SP.$$

If the periodic times are not the same,

$$F \text{ to } R \; : \; F \text{ to } S = \frac{SG^3}{P^2 \text{ round } R} \; : \; \frac{RP^2 . SP}{P^2 \text{ round } S}.$$

Prop. VIII. *To find the law of force by which a body may describe a semicircle, the center of force being so distant, that all lines drawn from it to the body may be considered parallel.*

Let *PQ* be a small arc of the semicircle, *C* the center; draw *PS, QS* parallel to each other toward the center of force; *CM* perpendicular to *PS*, then *CM* produced both ways will determine the semicircle described. Draw *QT* perpendicular, and *QR* parallel to *SP*, and produce *PR, TQ* to meet in *Z*; join *CP*. The triangles *PZT, CPM* are evidently similar;
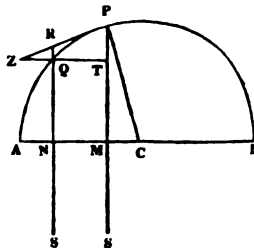
$$\therefore \ \frac{QR \cdot (RN + QN)}{QT^2} = \frac{RP^2}{QT^2} = \frac{ZP^2}{ZT^2} = \frac{CP^2}{PM^2};$$

$$\therefore \ \text{limit} \ \frac{QR}{QT^2} = \frac{CP^2}{2 PM^3}, \ \text{since limit} \ (RN + QN) = 2 PM;$$

$$\therefore \ F = \frac{2h^2}{SP^2} \cdot \text{limit} \ \frac{QR}{QT^2}$$

$$= \frac{h^2 \cdot CP^2}{SP^2 \cdot PM^3}, \ \text{or} = \frac{\mu}{PM^3}, \ \text{and} \ \therefore \propto \frac{1}{PM^3}.$$

Cor. To find the velocity at any point.

$$V^2 = F \cdot \frac{PV}{2} = \frac{h^2 \cdot CP^2}{SP^2 \cdot PM^3} \cdot PM$$

$$= \frac{h^2 \cdot CP^2}{SP^2 \cdot PM^2},$$

$$\therefore \ V = \frac{h \cdot CP}{SP \cdot PM}, \ \text{or} = \frac{\sqrt{\mu}}{PM}.$$

Scholium to Prop. VIII.

If AQP be any conic section, it may be described by the action of a force tending to a point at an infinite distance, and varying inversely as the cube of the ordinate.

Let PO, the diameter of curvature at *P,* cut the axis of the conic section in *K* ; draw *OV* perpendicular to *PS*, then *PV* is the chord of curvature at *P* in direction of the force; and complete the construction as in the proposition.
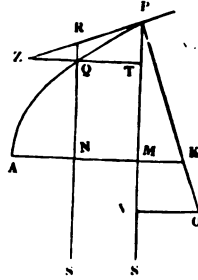
By similar triangles *ZPT*, *PMK*,

$$\frac{QT^2}{QR} \;:\; \frac{RP^2}{QR} = ZT^2 \;:\; ZP^2 = PM^2 \;:\; PK^2,$$

and this being true always will be true, when *Q* moves up to *P,*

$$\therefore \text{ L. R. } \frac{QT^2}{QR} \;:\; \frac{RP^2}{QR} = PM^2 \;:\; PK^2,$$

$$\text{and } PV \;:\; PO = PM \;:\; PK,$$

$$\therefore \text{ since limit } \frac{RP^2}{QR} = \text{limit } \frac{PQ^2}{QR} = PV,$$

$$\text{limit } \frac{QT^2}{QR} \;:\; PO = PM^3 \;:\; PK^3.$$

Now (Appendix Arts. 4, 5.) in all conic sections, the diameter of curvature $= \dfrac{8}{L^2}.PK^3,$

$$\therefore \text{ limit } \frac{QT^2}{QR} = \frac{8\,PM^3}{L^2},$$

$$\text{and } \therefore F = \frac{h^2.L^2}{4\,SP^2.PM^3} \varpropto \frac{1}{PM^3}.$$

PROP. IX.   *To find the law of force tending to the pole,*
by which a body may describe an equiangular spiral.

DEF.   An equiangular spiral, is a spiral cutting all the
radii at the same given angle.

Let PQ be a small arc
of the spiral, S the center
of force in the pole. QR a
subtense parallel to SP, QT
perpendicular to SP, and
let the constant angle SPR,
which the curve makes with
the radius, $= a$.   Join PQ, and let PV be the chord of curva-
ture through S,
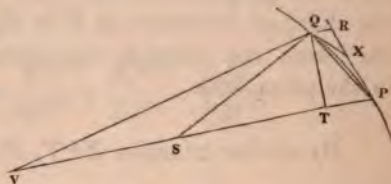
$$\text{then } \frac{QT^2}{QR} = \frac{PR^2}{QR} \sin^2 a ;$$

$$\therefore \text{ limit } \frac{QT^2}{QR} = \text{limit } \frac{PR^2}{QR} \sin^2 a = \text{limit } \frac{PQ^2}{QR} \sin^2 a = PV \sin^2 a.$$

Let the tangent at Q intersect PR in X.   Then since SP,
SQ make equal angles with the tangents at P, Q, the angles
SPX, SQX are equal to two right angles, therefore the angle
$PSQ =$ angle QXR.   Also since V is a point in the circum-
ference of the circle of curvature, the angles XPQ, XQP are
each ultimately equal to QVB.   Hence the angle QXR, and
therefore the angle QSP is ultimately double of the angle QVS,
therefore $\angle SQV$ is ultimately equal to $\angle SVQ$, or $SV = SQ$
ultimately $= SP$.   Hence $PV = 2 SP$,

$$\text{and } \therefore F = \frac{2 h^2}{SP^2} \text{ limit } \frac{QR}{QT^2}$$

$$= \frac{2h^2}{SP^2} \cdot \frac{1}{2\,SP\,\sin^2 \alpha}$$

$$= \frac{h^2}{\sin^2 \alpha} \cdot \frac{1}{SP^3}, \text{ or } = \frac{\mu}{SP^3}, \text{ and } \therefore \propto \frac{1}{SP^3}.$$
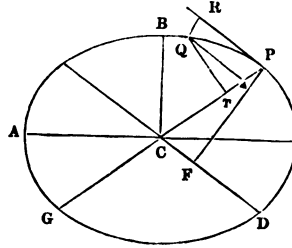
COR. To find the velocity at any point.

$$V^2 = F \cdot \frac{PV}{2} = \frac{h^2}{\sin^2 \alpha} \cdot \frac{1}{SP^2}, \text{ or } \frac{\mu}{SP^2};$$

$$\therefore V = \frac{h}{\sin \alpha} \cdot \frac{1}{SP}, \text{ or } \frac{\sqrt{\mu}}{SP}.$$

PROP. X. *A body describes an ellipse round a center of force in the center of the ellipse, to find the law of force.*

Let PQ be a small arc of the ellipse, C the center, QR a subtense parallel to CP; AC, BC the semi-axes major and minor; QV parallel to PR; QT, PF perpendicular to CP and the semi-conjugate CD respectively, produce PC to meet the ellipse again in G; then the triangles QVT, PCF are evidently similar.

Now $\dfrac{PV \cdot VG}{QV^2} = \dfrac{CP^2}{CD^2},$

and $\dfrac{QV^2}{QT^2} = \dfrac{CP^2}{PF^2};$

$\therefore \dfrac{PV \cdot VG}{QT^2} = \dfrac{CP^4}{PF^2 \cdot CD^2} = \dfrac{CP^4}{AC^2 \cdot BC^2};$

$$\therefore \ \text{limit} \ \frac{QR}{QT^2} = \text{limit} \ \frac{PV}{QT^2} = \frac{CP^4}{AC^2 . BC^2 . 2CP},$$

(since limit $VG = 2CP$)

$$= \frac{CP^3}{2AC^2 . BC^2};$$

$$\therefore \ F = \frac{2h^2}{CP^2} \ \text{limit} \ \frac{QR}{QT^2} = \frac{h^2}{AC^2 . BC^2} . CP,$$

or $= \mu . CP$, and therefore $\propto CP$.

Cor. 1. To find the velocity at any point.

$$V^2 = \tfrac{1}{2} F . PV = \tfrac{1}{2} \frac{h^2}{AC^2 . BC^2} \ CP . \frac{2CD^2}{CP}, \ \text{since} \ PV = \frac{2CD^2}{CP}$$

$$= \frac{h^2}{AC^2 . BC^2} \ CD^2 ;$$

$$\therefore \ V = \frac{h}{AC . BC} . CD, \ \text{or} \ \sqrt{\mu} . CD.$$

Cor. 2. To find the periodic time.

$$\text{Since} \ \mu = \frac{h^2}{AC^2 . BC^2}, \ h = AC . BC . \sqrt{\mu} ;$$

also the area of the ellipse $= \pi AC . BC$;

$$\therefore \ P = \frac{2 \ \text{area of ellipse}}{h}$$

$$= \frac{2\pi}{\sqrt{\mu}}.$$

Hence the periodic times in all ellipses round the same center of force in the center are equal.

Cor. 3.   *If a body be projected in a direction making any angle with its distance from a fixed point, and be attracted to that point by a force varying as the distance, it will describe an ellipse, whose center is the center of force.*

Let C be the center of force, P the point from which the body is projected in direction PY, V the velocity, and F the force at P.
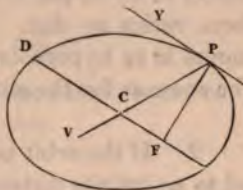
Then space (s) due to the velocity at $P = \dfrac{V^2}{2F}$. In PC, produced if necessary, take $PV = 4s$, and draw CD parallel to PY and $= \sqrt{\frac{1}{2}\,CP\,.\,PV}$. With CP, CD as semi-conjugate diameters describe an ellipse, and suppose a body revolving in it to come to P; then it is moving in the direction of the tangent at P, that is, in a line parallel to CD or in direction PY. Also space due to velocity at $P = \frac{1}{4}$ chord of curvature at P

$$= \frac{1}{4}\,.\,\frac{2\,CD^2}{CP} = \frac{1}{4}\,PV = s.$$

The force, distance, and law of force are the same also in both cases; hence the two bodies are under the same circumstances at P, and will therefore describe the same orbit; that is, the projected body will describe an ellipse, whose center is C.

If CPY be a right angle, and $s = \frac{1}{2}\,PC$, the orbit described will be a circle.

Cor. 4.   To compare the velocity at P with the velocity in a circle, radius $= CP$, described round the same center of force.

$$V. \text{ in ellipse} = \sqrt{\mu}\,.\,CD.$$

$$V. \text{ in circle (radius } = CP) = \sqrt{F\,.\,CP}, \quad \text{(Prop. vi. Cor. 5.)}$$

$$= \sqrt{\mu}\,.\,CP\,;$$

$$\therefore\ V. \text{ in ellipse} : V. \text{ in circle (rad.} = CP) = CD : CP.$$

F

Scholium to Prop. X.

1. It was proved in the proposition, that, when a body moves in an ellipse round a center of force in the center, the force varies as dist. The same is also true, when a body moves in an hyperbola, the construction and proof being exactly the same as for the ellipse.

2. If the orbit be a parabola, the center of force is removed to an infinite distance, and the force acts in lines parallel to the axis; in this case, since the difference of any two distances vanishes compared with the distances themselves, the force is invariable.

Or the following proof may be applied in the case of the parabola.

Let PQ be a small arc of the parabola, A the vertex, S the focus; PC parallel to the axis, and therefore in the direction of the force; QR a subtense parallel to PC, and QV parallel to the tangent PR; QT, SY perpendicular to CP, PR.
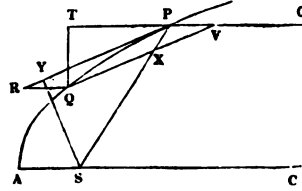
Since $4SP \cdot PV = QV^2$,
$$\frac{QR}{QV^2} = \frac{PV}{QV^2} = \frac{1}{4SP},$$

and by similar triangles, QTV, SPY,

$$\frac{QV^2}{QT^2} = \frac{SP^2}{SY^2} = \frac{SP}{SA};$$

$$\therefore \text{ limit } \frac{QR}{QT^2} = \frac{1}{4SA};$$

$$\therefore F = \frac{2h^2}{CP^2} \cdot \frac{1}{4SA}.$$

SECTION III.

ON THE MOTION OF BODIES IN CONIC SECTIONS, ABOUT A CENTER
OF FORCE IN ONE OF THE FOCI.

PROP. XI. *A body revolves in an ellipse, to find the law of force tending to one of the foci.*

Let the focus S be the center of force, PQ a small arc ; QR a subtense parallel to SP ; C the center of the ellipse, join PC and produce it to meet the ellipse in G ; draw Qxv parallel to the tangent PR, cutting SP, CP in x, v ; and QT, PF respectively perpendicular to SP, and the semi-conjugate diameter CD : and let E be the point, in which SP cuts CD, then $PE = AC$, the $\frac{1}{2}$ axis major.
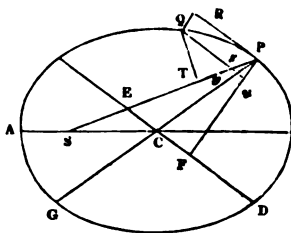
By similar triangles, QxT, PEF,

$$\frac{Qx^2}{QT^2} = \frac{PE^2}{PF^2} = \frac{AC^2}{PF^2},$$

and by a property of the ellipse,

$$\frac{Pv}{Qv^2} = \frac{CP^2}{vG \cdot CD^2} ;$$

also by similar triangles, PxV, PEC,

$$\frac{Px}{Pv} = \frac{PE}{PC} = \frac{AC}{PC}.$$

Now $Px = QR$, Qx ultimately $= Qv$, and vG ultimately $= 2CP$; hence multiplying the above quantities together, and taking the limits of the products,

$$\text{limit } \frac{QR}{QT^2} = \frac{AC^3 . CP}{2CP . CD^2 . PF^2} = \frac{AC^3}{2AC^2 . BC^2},$$

$$= \frac{AC}{2BC^2} = \frac{1}{L};$$

$$\therefore F = \frac{2h^2}{SP^2} \text{ limit } \frac{QR}{QT^2}$$

$$= \frac{2h^2}{L} \frac{1}{SP^2} \text{ or } = \frac{\mu}{SP^2}$$

$$\propto \frac{1}{SP^2}.$$

PROP. XII. *A body moves in an hyperbola, to find the law of force tending to one of the foci.*

Let the center of force be in the focus S, and let the body move in the branch PA, which is nearer to S than
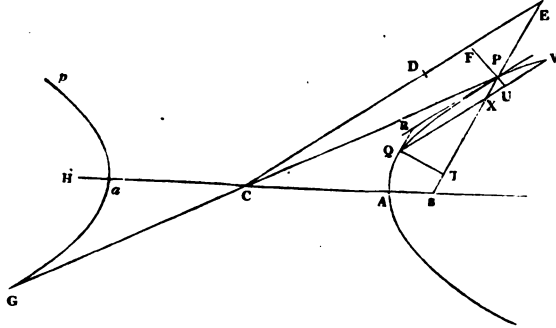

the other branch of the hyperbola. Then the same construction being made as in the ellipse, it may be shewn in precisely the same manner that the force

$$= \frac{2h^2}{L} . \frac{1}{SP^2} \text{ or } = \frac{\mu}{SP^2} \text{ and } \therefore \propto \frac{1}{SP^2}.$$

Cor. In the same manner it may be shewn, that if the body describes the opposite branch pa by a *repulsive* force proceeding from S, the force will vary as $\dfrac{1}{Sp^2}$.

PROP. XIII. *A body moves in a parabola, to find the law of force tending to the focus.*

Let A and S be the vertex and focus of the parabola, PQ a small arc; QR a subtense parallel to SP, QXV parallel to the tangent PR, cutting SP in X, and the diameter through P in V; QT, SY perpendicular to SP, PR; $L =$ latus rectum. Then by a property of the parabola $PV = PX$ and $\therefore\ = QR$; also $4SP \cdot PV = QV^2$.
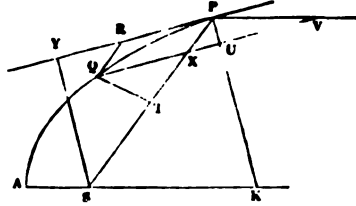
Hence

$$\frac{QR}{QV^2} = \frac{PV}{QV^2} = \frac{1}{4SP},$$

and by similar triangles,

$$\frac{QX^2}{QT^2} = \frac{SP^2}{SY^2} = \frac{SP}{SA}.$$

Now QX ultimately $= QV$; hence multiplying these quantities together, and taking the limits,

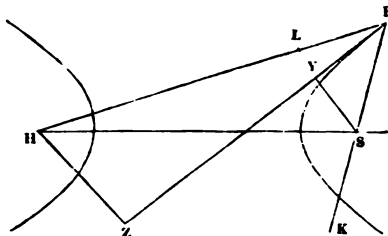$$\text{limit } \frac{QR}{QT^2} = \frac{1}{4SA} = \frac{1}{L},$$

$$\therefore\ F = \frac{2h^2}{SP^2} \text{ limit } \frac{QR}{QT^2},$$

$$= \frac{2h^2}{L} \cdot \frac{1}{SP^2} \text{ or } = \frac{\mu}{SP^2},$$
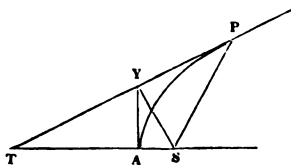
$$\propto \frac{1}{SP^2}.$$

Cor. *If a body be projected at a given distance from a center of force, which* $\propto$ *(dist.)*$^{-2}$, *and in a direction making a finite angle with the distance, it will describe a conic section.*

Let S be the center of force, P the point and PY the direction of projection, $F =$ the force at P, then if s be the space due to the velocity of projection, $s = \dfrac{(\text{velocity})^2}{2F}$ and is therefore known.

1. Let s be less than SP. In PS take $PK = s$, and draw PH, making with YP produced the $\angle HPZ = \angle SPY$; in PH take $PL = SK$, and through S, K, L describe a circle cutting PL in H; so that $PH \cdot PL = PS \cdot PK$. With foci S and H and axis major $= SP + HP$, describe an ellipse, and suppose a body revolving in this ellipse and acted on by the same force in S, to come to P; then space due to velocity at $P = \frac{1}{4}$ chord of curvature at P through S,

$$= \frac{(\frac{1}{2} \text{ conjugate diameter})^2}{\text{axis major}} = \frac{SP \cdot HP}{SP + HP},$$

$$= \frac{SP}{1 + \dfrac{SP}{HP}} = \frac{SP}{1 + \dfrac{PL}{PK}} = PK.$$

Hence the velocity is the same in both cases; also the revolving body is moving in direction PY, since ZPy, making equal angles with SP, HP, is a tangent at P; and the force and the law of force are the same for both bodies; they will therefore describe the same curve, that is, the projected body will describe an ellipse.

PROP. XV. *A body revolves in an ellipse round a center of force in the focus, to find the periodic time.*

Let AC, BC be the semi-axes major and minor, P the periodic time.

Then $\dfrac{P''}{1''} = \dfrac{\text{area of the ellipse}}{\text{area described in } 1''}$,

$\therefore P = \dfrac{\pi AC . BC}{\frac{1}{2} h}$,

and since $\dfrac{2 h^2}{L} = \mu$, $h = \sqrt{\dfrac{\mu L}{2}} = \sqrt{\dfrac{\mu . BC^2}{AC}} = BC \sqrt{\dfrac{\mu}{AC}}$,

$\therefore P = \dfrac{2 \pi AC^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}$.

COR. Hence, the squares of the periodic times in all ellipses, described round the same center of force in the focus, are as the cubes of the major axes.

PROP. XVI. *To find the velocity at any point of a conic section, described about a center of force in the focus.*

Let V be the velocity at the point P,

$$V^2 = \tfrac{1}{2} F . PV = \dfrac{\mu}{SP^2} . \dfrac{PV}{2}.$$

Now in the ellipse and hyperbola,

$$\dfrac{PV}{2} = \dfrac{CD^2}{AC} = \dfrac{SP . HP}{AC} = SP . \left(2 \mp \dfrac{SP}{AC} \right),$$

and in the parabola,

$$\dfrac{PV}{2} = 2 SP.$$

Hence in the ellipse $V = \sqrt{\dfrac{\mu}{SP}\left(2 - \dfrac{SP}{AC}\right)}$,

in hyperbola $V = \sqrt{\dfrac{\mu}{SP}\left(2 + \dfrac{SP}{AC}\right)}$,

in parabola $V = \sqrt{\dfrac{2\mu}{SP}}$.

Cor. To compare the velocity with that of a body moving in a circle, radius $= SP$, and described round the same centre of force.

Let $U =$ velocity in the circle,

then (Prop. vi. Cor. 5.),

$$U = \sqrt{F.R} = \sqrt{\dfrac{\mu}{SP^2}}\, SP. = \sqrt{\dfrac{\mu}{SP}},$$

$\therefore$ in ellipse $\dfrac{V}{U} = \sqrt{2 - \dfrac{SP}{AC}}$ which is less than $\sqrt{2}$,

in hyperbola $\dfrac{V}{U} = \sqrt{2 + \dfrac{SP}{AC}}$ .... greater ....
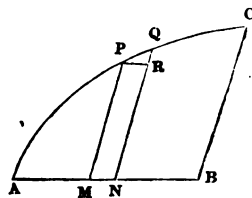
in parabola $\dfrac{V}{U} = \sqrt{2}$.

APPENDIX.

1.   To find the area of a plane curve.

Let the area *ABC* be bounded by the curve *AC*, and the straight lines *AB*, *BC*. Let *AB* be divided into *n* equal parts, and let *MN* be the rth part from *A*; draw *MP*, *NR* parallel to *BC*, and complete the parallelogram *MNRP*.

Let $AB = h$, then $MN = \dfrac{h}{n}$,

$MP = y_r$,

$\angle ABC = i$,

area of parallelogram $PN = \dfrac{h}{n} y_r \sin i.$

Therefore giving to *r* the values 1, 2, 3...*n*, the sum of the parallelograms described on all the parts

$$= \frac{h}{n} \sin i \; (y_1 + y_2 + y_3 + \dots + y_n) = h \sin i \cdot \Sigma \cdot \frac{y_r}{n}.$$

Therefore area of curvilinear figure $= h \sin i \cdot \text{limit } \Sigma \dfrac{y_r}{n}$, when *n* is infinite.

Ex. 1. To find the area of a portion of a parabola cut off by a diameter, and one of its ordinates.

Let ABC be the parabolic area cut off by the diameter AB and a semi-ordinate BC. Complete the parallelogram $ABCD$: then AD is a tangent at A.
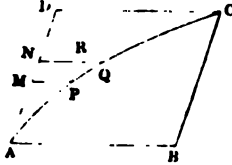
Let $AD = h$, $AB = k$, and let AM be the abscissa, and MP, parallel to AB, the ordinate to the point P; then by a property of the parabola,

$$\frac{PM}{AM^2} = \frac{AB}{AD^2} ; \quad \therefore PM \text{ or } y_r = \frac{k}{h^2} \cdot \left(\frac{rh}{n}\right)^2 = k \frac{r^2}{n^2},$$

$$\therefore \text{ area } ADC = h \sin i \cdot \text{ limit} \cdot \Sigma \frac{y_r}{n} = kh \sin i \text{ limit } \Sigma \frac{r^2}{n^3},$$

$$= hk \sin i \cdot \text{ limit} \cdot \frac{1}{n} \cdot (1^2 + 2^2 + 3^2 + \dots + n^2),$$

$$= hk \sin i \cdot \text{ limit } \frac{1}{n^3} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \tfrac{1}{3} hk \sin i$$

$$= \tfrac{1}{3} \text{ parallelogram } ABCD,$$

and $\therefore$ parabolic area $ABC = \tfrac{2}{3}$ circumscribing parallelogram.

2. The volume of a solid of revolution may be determined in a similar manner.

Let ABC be a plane curvilinear area by the revolution of which round AB the solid is generated, and let CB be perpendicular to AB. Then if AB ($= h$) be divided into n equal parts, and the rectangular parallelogram PN be described on MN the rth part, the cylinder generated by the revolution of PN round $MN = \frac{h}{n} \pi \cdot PM^2 = \frac{h}{n} \pi \cdot y_r^2,$

and the volume of the solid

$$= \text{limit . sum of all such cylinders}$$

$$= \pi h \,.\, \text{limit} \,\Sigma \,.\, \frac{y_r^{\,2}}{n}, \text{ when } n \text{ is infinite.}$$

Ex. 2.   To find the volume of a sphere.

Let **ABC** be a quadrant of the generating circle radius $= h$.

then $y^2 = 2hx - x^2$,

$$y_r^{\,2} = 2h\,\frac{rh}{n} - \left(\frac{rh}{n}\right)^2 = h^2\left(\frac{2r}{n} - \frac{r^2}{n^2}\right),$$

and therefore volume of hemisphere

$$= \pi h^3 \,\text{limit}\, \left\{\frac{2}{n^2}(1 + 2 + \ldots n) - \frac{1}{n^3}(1^2 + 2^2 + \ldots + n^2)\right\}$$

$$= \pi h^3 \,\text{limit}\, \left\{\frac{2}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) - \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right\}$$

$$= \pi h^3\left(1 - \frac{1}{3}\right) = \frac{2}{3}\pi h^3,$$

and therefore volume of sphere

$$= \frac{4}{3}\pi h^3 = \frac{2}{3}\,.\,2h\,.\,\pi h^2 = \frac{2}{3} \text{ circumscribing cylinder.}$$

Ex. 3.   Similarly the volume of a cone and paraboloid may be shewn to be $\frac{1}{3}$ and $\frac{1}{2}$ of the circumscribing cylinder respectively.

3.   To find the volume of a pyramid.

Let A be the area of the base of the pyramid, and let the perpendicular from the vertex upon the base $= h$. Divide h into n equal parts, and through the rth point of division from the vertex draw a plane parallel to the base. Then the area of the section of the pyramid thus made

$$= A\frac{\left(\dfrac{rh}{n}\right)^2}{h^2} = A\frac{r^2}{n^2};$$

On this area as a base describe a right prism, whose altitude $= \dfrac{h}{n}$; then volume of prism

$$= A\frac{r^2}{n^2}\cdot\frac{h}{n} = A h\frac{r^2}{n^3};$$

and therefore volume of pyramid = limit of sum of all such prisms

$$= A h \text{ limit } \Sigma\frac{r^2}{n^3} = A h \text{ limit } \frac{1}{n^3}\,(1^2 + 2^2 + \dots\dots + n^2)$$

$$= A h \text{ limit } \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \tfrac{1}{3} A h = \tfrac{1}{3}\text{ . base} \times \text{altitude}.$$

Note to Prop. XIII. on Curvature.

4.   To find the chords of curvature through the center and focus, and the diameter of curvature, at any point of an ellipse and hyperbola.   (Vide Figs. Props. xi. and xii.)

Let Qv, a semi-ordinate to the diameter PCG, cut SP in x, CP in v, and PF, which is perpendicular to the semi-conjugate CD, in u.

Chord of curvature through C

$$= \text{limit} \frac{PQ^2}{\text{subtense parallel to } CP}$$

$$= \text{limit} \frac{PQ^2}{Pv} = \text{limit} \frac{Qv^2}{Pv}$$

$$= \text{limit} \frac{CD^2}{CP^2} \cdot vG, \quad \text{since } \frac{Qv^2}{Pv \cdot vG} = \frac{CD^2}{CP^2}$$

$$= \frac{2CD^2}{CP}, \quad \text{since } vG \text{ ultimately} = 2CP.$$

Chord of curvature through S

$$= \text{limit} \frac{PQ^2}{\text{subtense parallel to } SP} = \text{limit} \frac{Qv^2}{Px}$$

$$= \text{limit} \frac{Qv^2}{Pv} \cdot \frac{Pv}{Px} = \text{limit} \frac{CD^2}{CP^2} \cdot vG \cdot \frac{PC}{PE}$$

$$= \frac{2CD^2}{AC}, \quad \text{since } PE = AC.$$

Diameter of curvature

$$= \text{limit} \frac{PQ^2}{\text{subtense perpendicular to tangent}}$$

$$= \text{limit} \frac{Qv^2}{Pu} = \text{limit} \frac{Qv^2}{Pv} \cdot \frac{Pv}{Pu} = \text{limit} \frac{CD^2}{CP^2} \cdot vG \cdot \frac{PC}{PF}$$

$$= \frac{2CD^2}{PF}.$$

Cor. Let PF cut the axis major in K, then $PF \cdot PK = BC^2$;

also $CD \cdot PF = AC \cdot BC$,

$\therefore$ diameter of curvature $= \dfrac{2AC^2 \cdot BC^2}{PF^3} = \dfrac{2AC^2 \cdot BC^2 \cdot PK^3}{BC^6}$

$= \dfrac{8PK^3}{L^2}$.

5. To find the chord of curvature through the focus, and the diameter of curvature at any point of a parabola. (Vide Fig. Prop. XIII.)

Let QV, a semi-ordinate to the diameter PV, cut SP in X; and the normal PK in U, draw SY perpendicular to the tangent at P; then $PX = PV$, hence

Chord of curvature through S

$= \operatorname{limit} \dfrac{PQ^2}{\text{subtense parallel to } SP}$

$= \operatorname{limit} \dfrac{PQ^2}{PX} = \operatorname{limit} \dfrac{QV^2}{PV}$

$= 4SP$, since $QV^2 = 4SP \cdot PV$.

Diameter of curvature

$= \operatorname{limit} \dfrac{PQ^2}{\text{subtense perpendicular to tangent}} = \operatorname{limit} \dfrac{QV^2}{PU}$

$= \dfrac{4SP \cdot PV}{PU} = 4SP \cdot \dfrac{SP}{SY}$, by sim^r. triangles PVU, SPY,

$= \dfrac{4SP^2}{SY}$, or $= 4\sqrt{\dfrac{SP^3}{SA}}$.

Cor.   Let PY meet the axis in T, then

$$ST = SP = SK, \quad \therefore SY = \tfrac{1}{2} PK ;$$

hence diameter of curvature

$$= 4 \, \frac{SP^2}{SY} = 4 \, \frac{SY^4}{SA^2 . SY} = \frac{4}{SA^2} \, SY^3 = \frac{1}{2 SA^2} . PK^2$$

$$= \frac{8 PK^3}{L^2} .$$

Note to Prop. VI. Cor. 3.

6.   If $SP = r$, and $SY = p$, $PV = \dfrac{2p}{d_r p}$.

(Miller's *Differential Calculus*. Art. 95.)

$$\therefore F = \frac{2 h^2}{SY^2 . PV} = \frac{2 h^2}{p^2 . \dfrac{2p}{d_r p}}$$

$$= \frac{h^2}{p^3} . d_r p.$$

Again, if $r = \dfrac{1}{u}$, $\dfrac{1}{p^2} = (d_\theta u)^2 + u^2$.

(Miller's *Differential Calculus*. Art. 89.)

$$\therefore \frac{1}{p^3} \, d_r p = - \frac{1}{p^3} . d_u p . u^2 = - \frac{1}{p^3} \, \frac{d_\theta p}{d_\theta u} . u^2$$

$$= (d_\theta u . d_\theta^2 u + u \, d_\theta u) \, \frac{u^2}{d_\theta u}$$

$$= u^2 (d_\theta^2 u + u) ;$$

$$\therefore F = h^2 u^2 (d_\theta^2 u + u).$$

Ex. 1. To find the law of force, by which a body may describe the curve $p = \dfrac{a\,r}{\sqrt{b^2 + r^2}}$, round a center of force in the pole.

$$\frac{1}{p^2} = \frac{b^2}{a^2 r^2} + \frac{1}{a^2},$$

and $\therefore \dfrac{1}{p^3}\,d_r p = \dfrac{b^2}{a^2 r^3};$

$$\therefore F = \frac{h^2}{p^3}\,d_r p = \frac{h^2 b^2}{a^2 r^3} \propto \frac{1}{r^3}.$$

Ex. 2. To find the law of force by which a body may describe a conic section, round a center of force at one extremity of the axis major.

Let S be the center of force at extremity of axis major SA, P any point in the curve, PN perpendicular to SA.

Let $SN = x, \quad PN = y,$

$\therefore y^2 = 2\,m\,x + n\,x^2$, is the equation to the curve.

Let $SP = r, \quad PSN = \theta;$

$\therefore r^2 \sin^2 \theta = 2\,m\,r \cos \theta + n\,r^2 \cos^2 \theta,$

$$r = \frac{2 m \cos \theta}{\sin^2 \theta - n \cos^2 \theta} = \frac{2 m \cos \theta}{1 - (1 + n) \cos^2 \theta};$$

$$\therefore u = \frac{1}{2 m} \left\{ \frac{1}{\cos \theta} - (1 + n) \cos \theta \right\},$$

$$d_\theta u = \frac{1}{2 m} \left\{ \frac{\sin \theta}{\cos^2 \theta} + (1 + n) \sin \theta \right\};$$

H

$$\therefore d_\theta^2 u = \frac{1}{2m} \left\{ \frac{1}{\cos\theta} + \frac{2\sin^2\theta}{\cos^3\theta} + (1+n)\cos\theta \right\},$$

$$d_\theta^2 u + u = \frac{1}{m} \left(\frac{1}{\cos\theta} + \frac{\sin^2\theta}{\cos^3\theta} \right)$$

$$= \frac{1}{m} \sec^2\theta,$$

$$\therefore F = h^2 u^2 (d_\theta^2 u + u)$$

$$= \frac{h^2}{m} \cdot \frac{\sec^3\theta}{r^2}, \quad \text{or} \propto \frac{SP}{SN^3}.$$

Note to Prop. X. Cor. 3.

6.   To find the magnitude and position of the axes of the orbit described.

Let $CP = r$, $CPy = a$, $s \left(= \dfrac{V^2}{2F} \right)$ = space due to velocity of projection, a and b the semi-axes of the orbit described.

$$CD = \sqrt{\tfrac{1}{2}\, CP \cdot PV} = \sqrt{2rs},$$

$$\left.\begin{array}{l} a^2 + b^2\ (= CP^2 + CD^2) = r^2 + 2rs \\[2mm] ab\quad (=\ CD \cdot PF)\ = \sqrt{2rs} \cdot r \sin a \end{array}\right\},$$

from which two equations a and b, and therefore e, the eccentricity, may be determined.

Also if θ be the inclination of axis major to CP,

$$r = \frac{b}{\sqrt{1 - e^2 \cos^2\theta}},$$

$$\therefore \cos\theta = \frac{1}{e} \sqrt{1 - \frac{b^2}{r^2}},$$

which may therefore be determined.

NOTE TO PROP. XIII. COR.

7. To find the magnitude and position of the axes of e orbit described.

Let $SP = r$, $\angle SPY = a$, draw SY, HZ perpendicular ᴐ YPZ; and let a, b, L be the semiaxes and latus rectum of ₁e orbit;

$$\text{then} \quad PL \cdot PH = PK \cdot PS,$$

$$\text{or} \quad (r \sim s) \cdot (2a \mp r) = s \cdot r,$$

$$\therefore \quad 2a(r \sim s) - r^2 = 0,$$

$$\therefore \quad a = \frac{r^2}{2(r \sim s)},$$

$$b = \sqrt{SY \cdot HZ} = \sqrt{SP \sin a \cdot HP \cdot \sin a}$$

$$= \frac{r \cdot s^{\frac{1}{2}} \cdot \sin a}{(r \sim s)^{\frac{1}{2}}};$$

$$\therefore \quad L = \frac{2b^2}{a}$$

$$= 4s \cdot \sin^2 a.$$

Again, $\qquad YZ = SH \cdot \sin ASY,$

$$\text{or} \quad (SP + HP) \cos a = 2e \cdot AC \sin ASY,$$

$$\therefore \quad \sin ASY = \frac{1}{e} \cos a,$$

hich equation, since $e = \sqrt{1 \mp \dfrac{b^2}{a^2}}$ is known, determines e position of the axis major.

ON ANGULAR VELOCITY.

8. The angular velocity of a body, moving in an orbit round a center of force S, (fig. page 28.) is measured by the angle uniformly described by SP round S in $1''$, in the same manner as linear velocity is measured by the line uniformly described in $1''$. If the angular motion of SP be not uniform, the angular velocity at any point is measured by the angle, which would be described in $1''$, if the angular motion of SP were to continue uniform for that time. Hence if the angular motion be not uniform, and PSQ be the angle described in T'' after leaving P, the angular velocity

$$= \text{limit } \frac{\text{angle } PSQ}{T},$$

for this is the angle which would be described in $1''$, if the angular motion at P were to continue uniform for that time.

PROP. *If a body be moving in any orbit round a center of force S, the angular velocity at any point P*

$$= \frac{h}{SP^2}.$$

Let PSQ be the angle described in T'', with center S and radius SQ, describe a circular arc cutting SP in T, and draw SY perpendicular to the tangent at P; then the triangle PTQ may be considered as ultimately rectilinear, and similar to SYP, hence

$$\angle^r \text{ vel. at } P = \text{limit } \frac{\angle PSQ}{T} = \text{limit } \frac{QT}{SQ \cdot T}$$

$$= \text{limit } \frac{PQ \cdot SY}{SP^2 \cdot T}, \text{ since limit } \frac{QT}{PQ} = \frac{SY}{SP}$$

$$= \frac{SY \cdot \text{vel. at } P}{SP^2}, \text{ since limit } \frac{PQ}{T} = \text{vel. at } P$$

$$= \frac{h}{SP^2}, \text{ (Prop. i. Cor. 3).}$$

9. *Force varying as (distance)$^{-2}$. To find the time of motion and the velocity acquired by a body falling through a given space from rest.* (PROPS. XXXIII. and XXXVI.)

Let S be the center of force, A the point from which the body begins to fall;

$$\frac{\mu}{SP^2} = \text{force at distance } SP.$$

Let APB be a semi-ellipse, focus S and axis major ASB; ADB a semi-circle, whose diameter is ASB; and suppose a body revolving in the ellipse round the focus S to come to P; bisect AB in O, draw DPC perpendicular to AB, and join OP, OD.
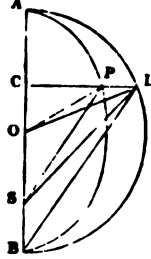
Then the time through $AP \propto$ area $ASP \propto$ area ASD, and this being true for all values of the axis minor will be true when it is diminished without limit, in which case the ellipse coincides with the axis major and the point P with C, or the body is moving in the straight line AC; the point B also coincides with S, since $AS . SB = (\frac{1}{2}$ axis minor$)^2$; and since space due to velocity at $A = \frac{1}{4}$ chord of curvature at A through $S = \frac{1}{4}$ latus rectum $= \dfrac{(\text{axis minor})^2}{4\,AB} = 0$, the body begins to move from rest at A.

Hence time from rest through $AC \propto$ area ABD,

$$\therefore \frac{\text{time rhrough } AC}{\text{time through } AB\,(=\frac{1}{2}\,\text{periodic time in ellipse})} = \frac{\text{area } ABD}{\text{semi-circle } ABD};$$

$$\therefore \text{ time through } AC = \frac{\pi . AO^{\frac{3}{2}}}{\sqrt{\mu}} . \frac{\frac{1}{2} AO . (AD + CD)}{\frac{1}{2} \pi . AO^2}$$

$$= \sqrt{\frac{AS}{2\mu}} . (AD + CD).$$

Again, velocity at $P = \sqrt{\dfrac{\mu}{AO} \cdot \dfrac{HP}{SP}}$ (Prop. XVI.) and when

the ellipse coincides with the axis major,

$$\text{velocity at } C = \sqrt{\frac{2\mu}{AS} \cdot \frac{AB - BC}{BC}} = \sqrt{\frac{2\mu}{AS} \cdot \frac{AC}{SC}}.$$

10. *Force varies as distance. To find the time of motion and the velocity acquired by a body in falling through a given space from rest.* (Prop. XXXVIII.)

Let S be the center of force, A the place from which the body begins to fall : on AB $= 2AS$ describe a semi-ellipse APB, and a semi-circle ADB, and let a body moving in the ellipse come to P. Draw DPC perpendicular to AB, and join SP, SD.
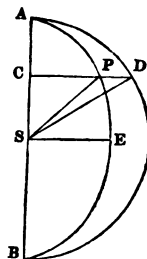
Then time through $AP \propto$ area ASP $\propto$ area ASD, and this being true, whatever be the axis minor of the ellipse, will be true when it is diminished without limit, in which case the body will be at C, having fallen from rest at A,

$\therefore$ time through $AC \propto$ area ASD

$\therefore \quad \dfrac{\text{time through } AC}{\text{time through } AS\, (= \frac{1}{4} \text{ periodic time in a circle})}$

$$= \frac{\text{sector } ASD}{\frac{1}{4} \text{ area of a circle}};$$

$\therefore$ time through $AC = \dfrac{\pi}{2\sqrt{\mu}} \cdot \dfrac{\frac{1}{2} AS \cdot AD}{\frac{1}{4}\pi\, AS^2}$

$$= \frac{AD}{AS\sqrt{\mu}}.$$

Again, let SE be the semi-axis minor,

then vel. at P = semi-conjugate at P . $\sqrt{\mu}$. (Prop. x. Cor. 1).

$$= \sqrt{AS^2 + SE^2 - SP^2} . \sqrt{\mu},$$

$\therefore$ vel. at $C = \sqrt{AS^2 - SC^2} . \sqrt{\mu}$

$$= CD\sqrt{\mu}.$$

11. *If the velocities of two bodies, one of which is falling directly towards a center of force, and the other describing a curve about that center, be equal at any equal distances, they will always be equal at equal distances.* (Prop. xl.)

Let S be the center of force, and let one of the bodies be moving in the straight line APS and the other in the curve AQq; with radii SQ, Sq describe the circular arcs QP, qp: let SQ cut pq in m, and draw mn perpendicular to Qq; and suppose the velocities of the bodies at P and Q to be equal.
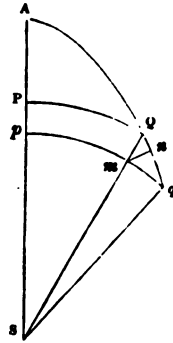
Since the centripetal forces at P and Q are equal, Pp, Qm may be taken to represent them: Pp is wholly effective in accelerating P, but the effective part of Qm is Qn, nm being wholly employed in retaining the body in the curve. Also since the velocities at P and Q are equal, the times of describing Pp and Qq, when the spaces are diminished indefinitely, are proportional to Pp and Qq; hence

$$\text{force at } P : \text{force at } Q = Pp : Qn$$

and time through Pp : time through $Qq = Pp : Qq$;

$\therefore$ velocity acquired at p : velocity acquired at q

$$= Pp^2 : Qn . Qq = Qm^2 : Qn . Qq$$

$$= 1 : 1,$$

and the same may be shewn at all corresponding points equally distant from S, therefore, *If the velocities, &c.*

ERRATA.

Page 1, in the last line but one, after "then" insert "the ratio of."
— 16 line 2, *for* RQr *read* RQq.
— 29 — 10, *for* chords *read* chord.
— 50 — 8, *for* NR *read* NQ.